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THE ELECTROMAGNETIC THEORY AND THE VELOCITY OF LIGHT.[1]

By Henry T. Eddy.

Introduction.

MR. MILLS has recently published a paper[2] in which he has given the results of measurements made by him of the increase in the velocity of circularly polarized light in bisulphide of carbon along the lines of force in a magnetic field. Employing circulary polarized light he was successful in obtaining a difference of one or more wave-lengths between two rays circularly polarized in opposite senses, one ray having its velocity increased while the other was decreased, and this was obtained with apparatus with which no difference whatever was observable in case of plane polarized rays.

The apparatus mentioned is a form of interferometer devised by Professor Morley and paid for by a grant made by the American Association for the Advancement of Science for the purpose of investigating certain points to which the present writer had taken exception in the theory of the Faraday effect as developed by Professor Rowland,[3] who had attempted to account for the twisting of the plane of polarization of plane polarized light while being propagated along the lines of force in a magnetically active medium by the action of the Hall effect in the medium. As

[1] Presented in abstract at the International Congress of Arts and Sciences at St. Louis, September, 1904.

[2] Physical Review, Vol. XVIII., p. 65, February, 1904.

[3] Am. Jour. Math., Vol. 3, p. 109, 1880.

just stated the present writer found himself unable to agree with that part of Professor Rowland's most valuable theoretical treatment of the Hall effect which related to rotary polarization. After a full presentation of the theoretical questions involved at the Toronto Meeting of the American Association for the Advancement of Science in 1889 the apparatus was constructed, and after many delays a final report was presented to the American Association for the Advancement of Science at the Boston meeting, August, 1898, That report was duly published [1] and contains first, the present writer's theoretical developments and computations as to the possible increase or decrease in velocity to be looked for in case of magnetic twisting of the plane of polarization, and second a full description by Professor Morley of his apparatus and a detailed account of the experimental work by Professors Morley and Miller who worked in collaboration. No experimental change in the velocity of plane polarized light could be detected with this apparatus, and the numerical computations just mentioned showed in fact the possible change in the velocity to be too minute to be detected by the apparatus as used. Although such is the fact with plane polarized light, the experiments of Mr. Mills show that such is not the fact with circularly polarized light. Moreover it will be shown theoretically that in case of circularly polarized light the amount of change in velocity due to the magnetic field is expressible as a lower power of small quantities, than in case of plane polarized light, and consequently the magnitude of the change in the former case is large compared with the latter, and in fact varies as the square of the latter; and while the latter may be quite beyond the range of observation, the former may be well within it as the experiments of Mr. Mills have proven.

In view of this, it is the aim of this paper in the first place to rediscuss the questions at issue and point out more in detail than heretofore how according to elementary theory the velocities of plane and circularly polarized rays in any optically or magnetically active medium must be necessarily related to each other and how according to elementary theory it is impossible that Professor Rowland's equations can represent a twisted plane polarized ray.

[1] PHYSICAL REVIEW, Vol. VII., p. 282, December, 1898.

In the second place it will be shown how these velocities in the magnetically active field are related to the velocity in zero field according to the several proposed hypotheses. Were it possible to make this comparison experimentally we should have a test as to the validity of the proposed hypotheses, but such test is as yet beyond reach by reason of the smallness of their differences.

An attempt is made in the third place to show that a moderate degree of absorption would exert a negligible influence in modifying the results already developed for perfectly transparent media.

I.

Let X and Y be proportional to displacements along x and y, then according to the undulatory theory of light the equations

$$\begin{aligned} X &= R \cos (mt - qz) \cos \theta, \\ Y &= R \cos (mt - qz) \sin \theta, \end{aligned} \tag{1}$$

together represent the transverse displacement in a wave propagated along z, this displacement lying in a line making an angle θ with x. If θ be constant the entire wave will lie in a plane inclined at an angle θ with x, and equations (1) will then represent plane polarized light and we have

$$\begin{aligned} &\text{the periodic time, } T = 2\pi/m, \\ &\text{the wave-length, } \lambda = 2\pi/q, \\ &\text{and the velocity, } v = m/q. \end{aligned} \tag{2}$$

But if we assume θ to vary uniformly along the path of the ray we have $\theta = pz$, in which p is constant, and equations (1) become

$$\begin{aligned} X &= R \cos (mt - qz) \cos pz, \\ Y &= R \cos (mt - qz) \sin pz, \end{aligned} \tag{3}$$

which represent a transverse displacement whose inclination to x uniformly increases as the wave advances along z, so that all displacements lie in a helical screw surface instead of lying in a plane as they do in (1).

This uniform twisting of the plane of polarization into a fixed helical surface is clearly what occurs in case of the Faraday effect where plane polarized light is propagated along the lines of force in a transparent magnetically active medium. If p be positive, the

plane of polarization is twisted into a right-handed screw, which makes one turn in a distance $\sigma = 2\pi/p$, provided the axes xyz be taken in such a relative position that the observer at the origin looking in the positive direction along z finds that a clockwise rotation of 90° will bring $+x$ to $+y$, and so on cyclically for the other axes.

It is self-evident that equations (3) represent without defect or excess the displacements occurring in the propagation of the twisted plane polarized wave in case of the Faraday effect due to a magnetic field; they are therefore the necessary and sufficient particular solutions of any differential equations having real coefficients only which express the propagation of light in general along the lines of force in a magnetically active medium. Let the transparent medium occupying the field be assumed to be isotropic. Then, according to the electromagnetic theory, displacement is in the direction and directly proportional to the strength of the electric field. Hence X and Y in (3) may be taken to express the strength of the electric field along x and y respectively, and the units in which they are measured will affect the numerical value of the constant, R, which has to do with the intensity of the light and the elasticity of the medium, which for the present is assumed to be perfectly transparent.

The periodic time in the twisted wave (3) is the same as in (1), but the wave-length and velocity, which we may call λ_0 and v_0 though still expressed by equations of the same form as (2) have values numerically different from λ and v in (2) because q is not the same in a magnetically active medium under stress of a magnetic field as it is at zero field.

By an obvious expansion equations (3) may be written in the form

$$\begin{aligned} X &= \tfrac{1}{2}R\cos\left[mt-(q-p)z\right]+\tfrac{1}{2}R\cos\left[mt-(q+p)z\right], \\ Y &= \tfrac{1}{2}R\sin\left[mt-(q-p)z\right]-\tfrac{1}{2}R\sin\left[mt-(q+p)z\right]. \end{aligned} \tag{4}$$

Now taking the axes of x, y, z in the right-handed order previously mentioned, the equations

$$\begin{aligned} X_1 &= \tfrac{1}{2}R\cos\left[mt-(q-p)z\right], \\ Y_1 &= \tfrac{1}{2}R\sin\left[mt-(q-p)z\right], \end{aligned} \tag{5}$$

represent a circularly polarized ray propagated along z which is right-handed to an observer at the origin, but clockwise to an observer looking towards the origin, and having the following periodic time, wave-length and velocity,

$$T_1 = \frac{2\pi}{m},\ \lambda_1 = \frac{2\pi}{q-p},\ v_1 = \frac{m}{q-p}. \tag{6}$$

Moreover the equations

$$\begin{aligned} X_2 &= \tfrac{1}{2}R\cos\left[mt-(q+p)z\right], \\ Y_2 &= -\tfrac{1}{2}R\sin\left[mt-(q+p)z\right], \end{aligned} \tag{7}$$

represent a left-handed, counter-clockwise, circularly polarized ray also propagated along z, in which we have

$$T_2 = \frac{2\pi}{m},\ \lambda_2 = \frac{2\pi}{q+p},\ v_2 = \frac{m}{q+p}. \tag{8}$$

By comparing (2) with (6) and (8) we have

$$T = T_0 = T_1 = T_2,\ \lambda_1 > \lambda_0 > \lambda_2,\ v_1 > v_0 > v_2, \tag{9}$$

and by comparing (4) with (5) and (7) we have

$$X = X_1 + X_2,\ Y = Y_1 + Y_2. \tag{10}$$

In other words a ray of plane polarized light uniformly twisted to the right may be regarded as composed of two component rays of the same color, a right-handed and a left-handed circularly polarized ray, the former having a greater and the latter a less velocity and wave-length than the twisted plane polarized ray, and since the latter velocity is known to differ from that of a ray in zero field by an amount too small to be observed, we may also write

$$\lambda_1 > \lambda > \lambda_2,\ v_1 > v > v_2. \tag{9}'$$

Fresnel[1] was able to show experimentally that a twisted plane polarized ray in an optically active medium actually consists of two opposite circularly polarized rays propagated with different velocities, their differing velocities and consequently unequal indices of refraction enabling him to separate the rays from each other by refraction.

[1] See Preston's Theory of Light, Longmans, 1890, pp. 361–2.

Professor Brace[1] was the first to show by a similar experiment that the velocity and index of refraction of circularly polarized light propagated along the lines of force in a magnetically active medium undergo a change due to the field such that the velocity is increased in case the circular displacement is in the same direction as the Amperian currents. And now Mr. Mills has confirmed this observation by experiment with the apparatus before mentioned.

By (6),

$$v_1 = \frac{m}{q}\left(1 - \frac{p}{q}\right)^{-1} = v_0\left(1 + \frac{p}{q} + \frac{p^2}{q^2} + \text{etc.}\right), \tag{11}$$

$$\therefore \frac{v_1 - v_0}{v_0} = \frac{p}{q}\left(1 + \frac{p}{q} + \text{etc.}\right), \tag{12}$$

which last expresses the ratio of the difference between the velocities of the plane and right circular rays to the velocity of the plane polarized ray.

By (8),

$$v_2 = \frac{m}{q}\left(1 + \frac{p}{q}\right)^{-1} = v_0\left(1 - \frac{p}{q} + \frac{p^2}{q^2} - \text{etc.}\right) \tag{13}$$

$$\therefore \frac{v_2 - v_0}{v_0} = -\frac{p}{q}\left(1 - \frac{p}{q} + \text{etc.}\right), \tag{14}$$

which last expresses what percentage the difference in the velocities of the left-nanded circularly polarized ray and the plane polarized ray is of the latter. Subtract (14) from (12)

$$\therefore \frac{v_1 - v_2}{v_0} = 2\frac{p}{q}\left(1 + \frac{p_2}{q_2} + \text{etc.}\right) \tag{15}$$

which shows what percentage the difference of the velocities of the right and left rays is of the velocity of the plane polarized ray. Let z_0, z_1, z_2 be the length of path occupied by the same number n of waves of the three kinds of polarized light respectively, twisted plane, right and left circular, then $z_0/\lambda_0 = z_1/\lambda_1 = z_2/\lambda_2 = n$.

But $v_0/\lambda_0 = v_1/\lambda_1 = v_2/\lambda_2 = m/2\pi$; therefore by (15)

$$\frac{v_1 - v_2}{v_0} = \frac{z_1 - z_2}{z_0} = 2\frac{p}{q} \text{ nearly.} \tag{16}$$

[1] Phil. Mag., April, 1901, p. 464.

But $q\lambda_0 = p\sigma = 2\pi$,

$$\therefore \frac{z_1 - z_2}{z_0} = \frac{2\lambda_0}{\sigma}, \tag{17}$$

Mr. Mills in his experiments had $z_0 = 300$ cm. in a field due to a current of about 13 amperes, and he found $\sigma = 600$ cm., and and $z_1 - z_2 = \lambda_0$. This result agrees perfectly with (17).

To compute the numerical value of $p/q = \lambda_0/\sigma$ for yellow light in bisulphide of carbon, taking the wave-length in air as 5.89×10^{-5} cm. and the index of refraction as 1.77, we have

$$\lambda_0 = 5.89 \times \frac{10^{-5}}{1.77} = \tfrac{1}{3} \times 10^{-4} \text{ nearly},$$

$$\therefore \frac{p}{q} = 5.55 \times 10^{-8} = \tfrac{1}{2} \times 10^{-7} \text{ nearly}.$$

Again by adding (12) and (14), etc.

$$\frac{\frac{1}{2}(v_1 + v_2) - v_0}{v_0} = \frac{p^2}{q^2}\left(1 + \frac{p^2}{q^2} + \text{etc.}\right), \tag{18}$$

$$\therefore \tfrac{1}{2}(v_1 + v_2) > v_0, \text{ or } v_1 - v_0 > v_0 - v_2; \tag{19}$$

i. e., the velocity of the twisted plane polarized ray is less than the arithmetical mean of the circularly polarized rays, or again the right-handed circularly polarized ray gains on the plane polarized ray faster than the plane ray gains on the left-handed ray.

The mathematical relations of the reciprocals of these velocities are somewhat more simple than of the velocities, for since

$$\frac{1}{v_1} = \frac{q - p}{m}, \quad \frac{1}{v_0} = \frac{q}{m}, \quad \frac{1}{v_2} = \frac{q + p}{m},$$

$$\therefore \tfrac{1}{2}\left(\frac{1}{v_1} + \frac{1}{v_2}\right) = \frac{1}{v_0}, \text{ and } \tfrac{1}{2}\left(\frac{1}{v_1} - \frac{1}{v_2}\right) = -\frac{p}{m};$$

i. e., the reciprocal of the velocity of the twisted plane polarized ray is the arithmetical mean of the reciprocals of the velocities of its component right and left circularly polarized rays.

The relative value of these velocities may be shown graphically as follows; since by equations (2), (6) and (8) we have

$$v_1(q - p) = v_0 q = v_2(q + p) = m.$$

Let m = twice the area of the triangle OBD of which the base $OB = q$ $\therefore$ the altitude $OD = v_0$. Let $OA = DC = p$, then the following areas are equal. $OCF = OAF \therefore OBD = OBC = ABF \therefore OF = v_2$. Again $AOC = AEC \therefore OBD = OGC = AGE \therefore OE = v_1$. Equations (19) etc. are seen to be true from this diagram.

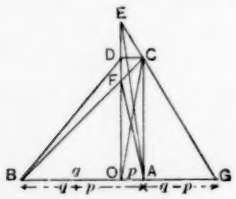

Fig. 1.

The deductions thus far are simple analytical (or we might say geometrical) consequences of the fundamental equations (3), joined with the experimental fact that a twisted plane polarized ray is actually resolvable into a right- and a left-handed circularly polarized ray, and they necessarily hold true whether the twisting be due to a magnetic field or to some other cause.

Now Professor Rowland in his paper previously referred to has proposed a pair of equations differing from (3) to express the propagation of a ray of twisted plane polarized light. His equations are to be derived from (1) by making θ vary uniformly with the time by writing $\theta = st$ in which s is a constant. Equations (1) then becomes

$$X = R \cos (mt - qz) \cos st,$$
$$Y = R \cos (mt - qz) \sin st. \tag{20}$$

It is self evident that these equations represent, not a twisted wave lying on a fixed helical surface such as happens in the Faraday effect, but instead of this a wave which at any given instant lies wholly in a single plane inclined at an angle $\theta = st$ to x but which as time passes rotates uniformly about z. Such a wave is not that contemplated in the Faraday effect, and it was assuredly incorrect to take equations (20) as representing a wave lying in a fixed helical surface. It will nevertheless be very useful to consider equations (20) somewhat in detail as we have already done in case of equations (3).

Equations (20) may be written in the form

$$X = \tfrac{1}{2}R \cos [(m + s)t - qz] + \tfrac{1}{2}R \cos [(m - s)t - qz],$$
$$Y = \tfrac{1}{2}R \sin [(m + s)t - qz] - \tfrac{1}{2}R \sin [(m - s)t - qz], \tag{21}$$

or,

$$X = X_1 + X_2, \text{ and } Y = Y_1 + Y_2;$$

if

$$\begin{aligned} X_1 &= \tfrac{1}{2}R \cos\,[(m+s)t - qz], \\ Y_1 &= \tfrac{1}{2}R \sin\,[(m+s)t - qz], \end{aligned} \tag{22}$$

and

$$\begin{aligned} X_2 &= \tfrac{1}{2}R \cos\,[(m-s)t - qz], \\ Y_2 &= -\tfrac{1}{2}R \sin\,[(m-s)t - qz]\,; \end{aligned} \tag{23}$$

from which it is evident that the uniformly rotating plane wave (20) may be regarded as consisting of a right-handed circularly polarized ray (22) and a left-handed ray (23) in which we have

$$\begin{aligned} T_1 &= \frac{2\pi}{(m+s)}, \quad \lambda_1 = \frac{2\pi}{q}, \quad v_1 = \frac{(m+s)}{q}, \\ T_2 &= \frac{2\pi}{(m-s)}, \quad \lambda_2 = \frac{2\pi}{q}, \quad v_2 = \frac{(m-s)}{q}. \end{aligned} \tag{24}$$

Equations (24) show that were it possible in any way, mechanically, optically, magnetically or otherwise, to produce and propagate such a ray as is expressed by (20), then the two opposite circularly polarized rays of which it might by (21) be supposed to be compounded would have different periodic times (or colors), and different velocities, but the same wave-length, the right-handed ray having the higher velocity in this case also.

Moreover we should have by (24)

$$\tfrac{1}{2}(v_1 + v_2) = \frac{m}{q} = v_0, \quad v_1 - v_0 = v_0 - v_2 = \frac{s}{q}; \tag{24}'$$

i. e., the velocity of light in this rotating plane wave is the arithmetical mean of the velocities of its right- and left-handed component rays, on which the right-handed wave gains as fast as the left-handed loses. Whether it would be possible actually to produce such a rotating plane wave as this, or whether if produced it could be separated experimentally into two opposite circularly polarized waves of different colors, it is impossible to say. It, however, seems highly improbable.

It is to be particularly noticed that in the discussion thus far none of the equations relating to the velocities of the magnetically twisted

or rotating plane polarized rays or their components have expressed any relation of these velocities to the velocity of the ray (polarized or not) in the same medium at zero field. Whether the effect of the field should be to increase the velocity of plane polarized light, or to retard it, can be known only by experiment. It may, in fact, happen that some media such as solids, for example, have an influence upon the velocity opposite to that of liquids or vapors. The differential equations which have been derived from several physical hypotheses to express the propagation of light in general in a magnetically active field give mathematical relations between the velocity at zero field and the other magnitudes involved in the wave motion, and they may be made to determine whether according to each hypothesis the velocity is increased or decreased by the field and by how much. Unfortunately, the amount of change is too minute for detection, so far, but since one hypothesis gives rise to equations that would make the field decrease the velocity, *i. e.*, make $v_0 < v$, and another would increase it so that $v_0 > v$, future success in the detection of the sign of the change would decide as between these two hypotheses. To the discussion of these equations we now proceed.

II.

In the paper previously referred to,[1] Professor Rowland has derived a pair of linear partial differential equations of the third order to express the propagation of light along the lines of force in a magnetically active medium under the influence of the cross currents of the Hall effect. In case of monochromatic light they may be written in the following form :

$$\begin{aligned} \frac{\partial^2 X}{\partial t^2} &= v^2 \frac{\partial^2 X}{\partial z^2} + f \frac{\partial^3 Y}{\partial t \partial z^2}, \\ \frac{\partial^2 Y}{\partial t^2} &= v^2 \frac{\partial^2 Y}{\partial z^2} - f \frac{\partial^3 X}{\partial t \partial z^2}. \end{aligned} \tag{25}$$

These equations were under discussion by me in the original report.[2] Now, these are precisely the differential equations which

[1] Am. Jour. Math., Vol. III., 1880.

[2] PHYSICAL REVIEW, Vol. III., 1898.

Drude[1] has shown to hold on the hypothesis of ionic charges in orbital motion in a magnetic field, and not as Professor Rowland states, on the hypothesis of the Hall effect. Drude has also derived another different pair of linear differential equations on the hypothesis of the controlling influence of the Hall effect, which will be considered later in this paper.

Equations (25) have been derived, then, to express the propagation of monochromatic light of any kind having the velocity v in zero field in a perfectly transparent medium as it moves in the direction of the lines of force, provided the medium itself has no natural period of vibration near that of the light in question. The final terms in these equations take account of the influence of the orbital motions of the ionic charges, and f is a small quantity proportional to the strength of the field. In case $f = 0$, equations (25) reduce to the partial differential equations of the second order which hold for light in ordinary isotropic transparent media, of which equations (1) would be particular solutions.

Since the pairs of equations (3), (5) and (7) each represent the propagation of a particular kind of light along the lines of force in a magnetically active medium in the direction of z, each of these pairs forms a particular solution of any pair of differential equations having real coefficients only which expresses the propagation of monochromatic light of any kind along z; they must therefore be particular solutions of (25), if (25) be valid. First substitute in (25) the particular values X_1 and Y_1 as given in (5), and we find after discarding a common factor one and the same equation from either of equations (25), viz.:

$$m^2 = (q - p)^2 (v^2 + mf) \qquad (26)$$

which gives the relation which must exist between the constants $m\ p\ q$ of (5) and $v\ f$ of (25) when (5) is a solution of (25).

It is to be noticed that this one equation of relation (26) is sufficient to determine the constant f supposing $m\ p\ q$ and v are known experimentally.

From (26) we have

$$v_1 = \frac{m}{q-p} = v\left(1 + \frac{mf}{v^2}\right)^{\frac{1}{2}} = v\left(1 + \tfrac{1}{2}\frac{mf}{v^2} - \tfrac{1}{8}\frac{m^2f^2}{v^4} + \text{etc.}\right); \quad (27)$$

[1] Theory of Optics, Longmans, 1903, p. 426.

$$\therefore \frac{v_1 - v}{v} = \tfrac{1}{2}\frac{mf}{v^2}\left(1 - \tfrac{1}{4}\frac{mf}{v^2} + \text{etc.}\right), \tag{28}$$

which last equation expresses the fraction which the increment of the velocity of the right-handed circularly polarized ray due to the action of the field is of the velocity at zero field.

Let $z = n\lambda$ be any distance in the medium occupied by some number n of wave-lengths in zero field, and $z_1 = n\lambda_1$ be the distance in the medium occupied by the same number of wave-lengths in the given field;

$$\therefore \frac{z_1 - z}{z} = \frac{\lambda_1 - \lambda}{\lambda} = \frac{v_1 - v}{v}, \tag{29}$$

if $v = \lambda/T$ and $v_1 = \lambda_1/T$.

Hence by (28) and (29) we have approximately

$$\frac{z - z_1}{z} = \tfrac{1}{2}\cdot\frac{mf}{v^2}, \tag{30}$$

from which f may be computed for a given field, for a given medium, and for light of a given period $T = 2\pi/m$, when the distance z has been observed in which a change of one wave-length $z - z_1 = \lambda$, for example, occurs, as was the case in the experiment of Mr. Mills, who found that a right-handed ray gained a wave-length on a left-handed ray in 300 cm. in a field produced by a current of 13 amperes. As a first approximation a right handed ray in the field may be taken to gain a wave-length on a ray in zero field in 600 cm. $= z$. If the wave-length of sodium light be taken at 5.89×10^{-5} cm. in air, and the index of refraction for bisulphide of carbon as 1.77, then in the latter medium $\lambda = 5.89 \times 10^{-5}/1.77 = \frac{1}{3} \times 10^{-4}$ nearly.

Hence by (30)

$$\tfrac{1}{2}\frac{mf}{v^2} = \frac{\lambda}{600} = \tfrac{1}{2} \times 10^{-7}, \tag{31}$$

and if $v = 3 \times 10^{10}/1.77$, and $m = 2\pi/T = 2\pi v/\lambda$ we obtain

$$f = \frac{v\lambda^2}{600\pi} = 10^{-2} \text{ nearly}, \tag{31}'$$

from which it appears that in the field, etc., of these experiments the value of f is so small that the expansions (27) and (28) are in terms

of ascending powers of a very small quantity, so that the approximation of (30) is justified.

Again on substituting for X and Y in (25) the particular values X_2 and Y_2 given in (7) for a left-handed circularly polarized ray we get the same equations resulting from both of equations (25), viz :

$$m^2 = (q + p)^2(v^2 - mf), \tag{32}$$

which should give the same numerical value of f in terms of m p q and v as (26).

By (32),

$$v_2 = \frac{m}{q+p} = v\left(1 - \frac{mf}{v^2}\right)^{\frac{1}{2}} = v\left(1 - \tfrac{1}{2}\frac{mf}{v^2} - \tfrac{1}{8}\frac{m^2f^2}{v^4} - \text{etc.}\right), \tag{33}$$

$$\therefore \frac{v_2 - v}{v} = -\tfrac{1}{2}\frac{mf}{v^2}\left(1 + \tfrac{1}{4}\frac{mf}{v^2} + \text{etc.}\right), \tag{34}$$

from which it is to be seen that the decrease in velocity of this left-handed ray is very approximately the same numerically as the increase in the velocity of the right-handed ray, but not identically the same, the decrease being the smaller, for by (28) and (34) we have

$$\frac{\frac{1}{2}(v_1 + v_2) - v}{v} = -\tfrac{1}{8}\frac{m^2f^2}{v^4}, \text{ or } v_1 - v > v - v_2, \tag{35}$$

from which it appears that the right-handed ray in the field gains on a ray in zero field faster than the latter gains on a left-handed ray in the field.

If now, once more, the particular values of X and Y for twisted plane polarized light given in (3), or better in (4), be substituted in each of equations (25) the two results thus obtained are not identical as was found to be the case for each of the circularly polarized rays separately. If for brevity we write

$$mt - (q - p)z = \varphi_1, \text{ and } mt - (q + p)z = \varphi_2, \tag{36}$$

we find by this substitution the equations

$$\begin{aligned} m^2(\cos\varphi_1 + \cos\varphi_2) &= (q-p)^2(v^2+mf)\cos\varphi_1 + (q+p)^2(v^2-mf)\cos\varphi_2, \\ m^2(\sin\varphi_1 - \sin\varphi_2) &= (q-p)^2(v^2+mf)\sin\varphi_1 - (q+p)^2(v^2-mf)\sin\varphi_2. \end{aligned} \tag{37}$$

In order that these equations may hold for all values of φ_1 and φ_2 the coefficients of the sines and cosines of both φ_1 and φ_2 in (37) must separately vanish. So that we have in this case either that (26) and (32) hold simultaneously, or what is the same thing (27) and (33), or (28) and (34). With two equations of relation between the quantities $m\ p\ q\ f\ v$ it is possible to determine f and v in terms of the given constants $m\ p\ q$, and compare v with its experimental value as a test of the validity of (25). Owing to experimental difficulties in making the comparison with sufficient accuracy we shall develop the formulæ for comparison in a somewhat different form. Subtract (32) from (26) then

$$(q^2 + p^2)mf = 2qpv^2;$$

$$\therefore \frac{mf}{2v^2} = \frac{p}{q}\left(1 + \frac{p^2}{q^2}\right)^{-1} = \frac{p}{q}\left(1 - \frac{p^2}{q^2} + \text{etc.}\right) = \frac{p}{q}\text{ nearly} \qquad (38)$$

from which it is evident that p and f are of essentially the same sign. Equations (38) are of fundamental importance in this theory as they express most directly the relation of the undetermined constant f in (25) to the experimental constant $p = \theta/z$ where θ is the amount of twisting in the distance z. The value of f thus obtained from the amount of twisting must of necessity be numerically equal to that obtained by (31) from the change of velocity of circularly polarized light due to the field and would agree practically with that obtained from the computation following equations (16) and (17).

Again by help of (38) we may write (26) and (32) as follows:

$$\begin{aligned} m &= v(q-p)\left(1 + \frac{2p}{q}\right)^{\frac{1}{2}} = v(q-p)\left(1 + \frac{p}{q} - \tfrac{1}{2}\frac{p^2}{q^2} + \text{etc.}\right), \\ m &= v(q+p)\left(1 - \frac{2p}{q}\right)^{\frac{1}{2}} = v(q+p)\left(1 - \frac{p}{q} - \tfrac{1}{2}\frac{p^2}{q^2} - \text{etc.}\right). \end{aligned} \qquad (39)$$

Add these equations, etc., and we find:

$$v_0 = \frac{m}{q} = v\left(1 - \tfrac{3}{2}\frac{p^2}{q^2} - \text{etc.}\right),$$

$$\therefore \frac{v_0 - v}{v} = -\tfrac{3}{2}\frac{p^2}{q^2}\text{ nearly, or } v_0 < v, \qquad (40)$$

which equations show that according to (25) the velocity $v_0 = m/q$ of the twisted plane wave (3) is less than the velocity v in zero field, *i. e.*, the velocity is slightly decreased by the twisting due to the field, and they show the amount of decrease. By (38) we may write (27) in the form

$$v_1 = v\left(1 + \frac{p}{q} - \tfrac{1}{2}\frac{p^2}{q^2} + , \text{etc.}\right), \tag{41}$$

and if we divide (41) by (40) we shall reproduce (11); and similarly, if we divide (33) when written in terms of p/q according to (38) by (40) we shall reproduce (13). These results are a consequence of the fact that equations (4) constitute a particular solution of (25).

Equation (40) was employed in the original report before mentioned to compute the amount of the decrease to be looked for in the velocity of twisted plane polarized light by reason of the magnetic field in the following manner. As stated previously,

$$p\sigma = 2\pi = q\lambda \quad \therefore \quad \lambda/\sigma = p/q, \tag{42}$$

in which σ is the distance along z in which the twisted plane-wave makes one turn, and again since

$$\frac{v_0 - v}{v} = \frac{z_0 - z}{z}, \tag{43}$$

in case $z/\lambda = n = z_0/\lambda_0$ is the number of wave-lengths in a distance z_0 in the given field and in a distance z in zero field, then by (40) and (42),

$$\frac{\delta z_0}{\lambda} = \frac{z_0 - z}{\lambda} = -\tfrac{3}{2}\frac{\lambda z}{\sigma} \tag{44}$$

which expresses what fraction of a wave-length a twisted plane wave would lose in traversing a distance z in the field. There is a correction which should be made in the numerical computation in the original report, and that is to replace the approximate value of the wave-length of sodium light in air, by that in bisulphide of carbon, *i. e.*, if the former be taken to be 6×10^{-5} cm., then the latter would be nearly $\lambda = 10^{-5} \times 6/1.77 = \frac{1}{3} \times 10^{-4}$ cm., so that if we take the figures in the report, viz. $z = 120$ cm. and $\sigma = 240$ cm., we find approximately $\delta z_0/\lambda = -10^{-7}$, $\delta z_0 = -\frac{1}{3} \times 10^{-11}$, $\delta z_0/z = -0.3 \times 10^{-13}$ as the corrected results, which would leave the conclusions before enunciated entirely unchanged.

Let us next consider Professor Rowland's equations (20), or (21), and their components as particular solutions of his differential equations (25). We have already learned that circularly polarized right- or left-handed rays are particular solutions of (25), therefore (22) and (23) separately satisfy (25), and proceeding to substitute (21) we find the following equations for this case in strict analogy with (26) and (32) before

$$\begin{aligned}(m+s)^2 &= q^2[v^2+(m+s)f],\\ (m-s)^2 &= q^2[v^2-(m-s)f],\end{aligned} \tag{45}$$

$$\therefore v_1 = \frac{m+s}{q} = v\left[1+\frac{(m+s)}{v^2}f\right]^{\frac{1}{2}},$$

and

$$v_2 = \frac{m-s}{q} = v\left[1-\frac{(m-s)}{v^2}f\right]^{\frac{1}{2}}; \tag{46}$$

or

$$v_1 = v\left(1+\tfrac{1}{2}\frac{(m+s)}{v^2}f-\tfrac{1}{8}\frac{(m+s)^2}{v^4}f^2+\text{etc.}\right),$$

and

$$v_2 = v\left(1-\tfrac{1}{2}\frac{(m-s)}{v^2}f-\tfrac{1}{8}\frac{(m-s)^2}{v^4}f^2-\text{etc.}\right). \tag{47}$$

Take the difference of equations (45), etc.,

$$\therefore s = \tfrac{1}{2}q^2f = \tfrac{1}{2}\frac{m^2f}{v^2}\text{ nearly.} \tag{48}$$

By (46), (47) and (24)′

$$\tfrac{1}{2}(v_1+v_2) = v_0 = \frac{m}{q} = v\left(1+\frac{sf}{2v^2}-\frac{(m^2+s^2)}{8v^4}f^2+\text{etc.}\right), \tag{49}$$

therefore by (48),

$$v_0 = v\left(1+\frac{m^2f^2}{8v^4}+\text{etc.}\right), \tag{50}$$

$$\therefore \frac{v_0-v}{v} = \frac{m^2f^2}{8v^4},\ \therefore v_0 > v; \tag{51}$$

from which it appears that a rotating plane-wave would have its velocity increased by the field.

By the help of (48), equations (47) become

$$v'_1 = v'\left(1 + \frac{mf}{2v'^2} + \frac{m^2f^2}{8v'^4} \pm \text{etc.}\right),$$
$$v'_2 = v'\left(1 - \frac{mf}{2v'^2} + \frac{m^2f^2}{8v'^4} \pm \text{etc.}\right), \tag{52}$$

which equations may be reproduced by multiplying (50) by equations (24) written in the following form,

$$v'_1 = \frac{m}{q}\left(1 + \frac{s}{m}\right) = v_0\left(1 + \frac{mf}{2v'^2}\right),$$
$$v'_2 = \frac{m}{q}\left(1 - \frac{s}{m}\right) = v_0\left(1 - \frac{mf}{2v'^2}\right). \tag{53}$$

This result is a consequence of using (21) as a solution of (25) The approximate equation

$$v'_1 - v'_2 = \frac{mf}{v'} \tag{54}$$

may be derived from equations (27) and (33), or from equations (52), it being consequently the difference of the velocities of the circularly polarized components for both the cases, a twisted and a rotating plane polarized ray, but it is not certain that the value of f would be precisely the same in both cases, provided it were possible to produce the rotating plane rays. Assuming, however, the identity of f in the two cases, the increase to be looked for in (51) is one third the decrease in (40), as may be seen by (38).

Drude has shown[1] that while equations (25) are in close agreement with many optical phenomena of magnetically active media, nevertheless certain experimental results in case of gases and vapors are not in accord with these equations founded on the hypothesis of orbital motions of ionic charges. He has, therefore, investigated the hypothesis of the cross currents due to the Hall effect and given various experimental results confirmatory of the truth of this hypothesis for vapors, especially of sodium. It seems desirable, therefore, to consider the results of this second hypothesis at some length, for it is as yet unknown which of these may better account for the phenomena in case of liquids, in which the accepted theory of solution favors the view that ionic charges may have much the same

[1] Theory of Optics, Longmans, 1903, Ch. VII.

mobility as in gases and vapors. It may be, in fact, true that both kinds of action coexist in liquids, for since the first hypothesis is probably more nearly true in solid dielectrics, and the second in vapors, we may readily believe that convective cross currents exist in liquids of sufficient magnitude to give rise to effects similar to those in vapors.

The differential equations of the Hall effect may be written,

$$\begin{aligned} \frac{\partial^2 X}{\partial t^2} &= v^2 \frac{\partial^2 X}{\partial z^2} + if \frac{\partial^2 Y}{\partial t^2}, \\ \frac{\partial^2 Y}{\partial t^2} &= v^2 \frac{\partial^2 Y}{\partial z^2} - if \frac{\partial^2 X}{\partial t^2}, \end{aligned} \tag{55}$$

which are assumed to hold for perfectly transparent media and for monochromatic light waves of a period remote from any natural periods of vibration of the media. In case this assumption is untrue, f becomes complex, and the first member of each of equations (55) will then have a complex coefficient.

The fact that the last term of each of equations (55) contains the imaginary factor i requires that the solutions shall also contain imaginaries, and the form of the solution to represent a twisted plane polarized wave will not be of the simple real form in equations (3) but the more general complex form of which (3) is the real part. This form is

$$\begin{aligned} X &= R[\cos(mt - qz) + i \sin(mt - qz)] \cos pz, \\ Y &= R[\cos(mt - qz) + i \sin(mt - qz)] \sin pz. \end{aligned} \tag{56}$$

By obvious expansions this may be written as the sum of two opposite circularly polarized rays

$$X_1 + X_2 = X \quad \text{and} \quad Y_1 + Y_2 = Y$$

in which if we employ the abbreviations (36) we have

$$X_1 = \tfrac{1}{2}R(\cos\varphi_1 + i \sin\varphi_1), \; Y_1 = \tfrac{1}{2}R(\sin\varphi_1 - i\cos\varphi_1), \tag{57}$$

$$X_2 = \tfrac{1}{2}R(\cos\varphi_2 + i \sin\varphi_2), \; Y_2 = -\tfrac{1}{2}R(\sin\varphi_2 - i\cos\varphi_2). \tag{58}$$

The real parts of equations (57) and (58) represent a right-handed and a left-handed circularly polarized ray, the one with a greater velocity and the other with a less velocity than v, just as in equations (4).

Equations (57) on being taken as a particular solution of equations (55) give rise to the single equation

$$m^2 = v^2(q-p)^2 + fm^2, \tag{59}$$

and similarly equations (58) give a single equation

$$m^2 = v^2(q+p) - fm^2. \tag{60}$$

Solving (59) and (60) we find

$$v_1 = \frac{m}{q-p} = v(1-f)^{-\frac{1}{2}} = v(1+\tfrac{1}{2}f+\tfrac{3}{8}f^2+\text{etc.}), \tag{61}$$

$$v_2 = \frac{m}{q+p} = v(1+f)^{-\frac{1}{2}} = v(1-\tfrac{1}{2}f+\tfrac{3}{8}f^2-\text{etc.}); \tag{62}$$

$$\therefore \frac{v_1-v}{v} = \tfrac{1}{2}f(1+\tfrac{3}{4}f+\text{etc.}), \tag{63}$$

and

$$\frac{v_2-v}{v} = -\tfrac{1}{2}f(1-\tfrac{3}{4}f+\text{etc.}). \tag{64}$$

Now, just as in the previous hypothesis, we are warranted in assuming that (59) and (60) coexist in case of a twisted plane polarized ray in a magnetic field.

Subtract (59) from (60) and we have

$$2pqv^2 = fm^2, \tag{65}$$

from which it is evident that f and p have essentially the same sign.

Now from (61) and (62) we find

$$\frac{1-f}{1+f} = \left(\frac{q-p}{q+p}\right)^2 = \frac{1-\dfrac{2qp}{q^2+p^2}}{1+\dfrac{2qp}{q^2+p^2}},$$

$$\therefore f = \frac{2qp}{q^2+p^2} = \frac{2\dfrac{p}{q}}{1+\dfrac{p^2}{q^2}} = \frac{2p}{q} \text{ nearly.} \tag{66}$$

Substituting this value of f in either (61) or (62) we have the approximate value of v_0 as far as to terms in f^2, viz:

$$v_0 = \frac{m}{q} = v\left(1 - \frac{p}{q}\right)\left(1 + \frac{p}{q}\right) + \tfrac{3}{2}\frac{p^2}{q^2} = v\left(1 + \frac{p^2}{q^2}\right); \qquad (67)$$

$$\therefore \frac{v_0 - v}{v} = \tfrac{1}{2}\frac{p^2}{q^2},$$

or

$$\frac{z_0 - z}{z} = \tfrac{1}{2}\frac{\lambda^2}{\sigma^2}, \qquad (68)$$

which shows that according to this hypothesis the velocity of plane polarized light is slightly increased as it is twisted in passing along the lines of force in a magnetic field, regardless of course of the sign of p and f, since p appears in (67) squared, as does f also, by (66).

A comparison of equations (40) and (68) shows that the consequences of the two hypotheses as developed from (25) and (55) differ in this that according to (25) a twisted plane polarized ray has its velocity decreased by the field by an amount which can be evaluated from the wave-length, twist, etc., while it follows in a similar manner from (55) that the ray has its velocity increased by the same field by an amount one third as great as the decrease just mentioned. In case these actions should coexist they would partially mask each other.

Besides the differential equations (25) and (55) derived from the electromagnetic theory there is also a pair of differential equations of the third order by which MacCullagh[1] first connected twisted plane polarization with the properties of transparent media. They may be written

$$\begin{aligned} \frac{\partial^2 X}{\partial t^2} &= v^2\frac{\partial^2 X}{\partial z^2} - f\frac{\partial^3 Y}{\partial z^3}, \\ \frac{\partial^2 Y}{\partial t^2} &= v^2\frac{\partial^2 Y}{\partial z^2} + f\frac{\partial^3 X}{\partial z^3}. \end{aligned} \qquad (69)$$

If (5) and (7) be taken as separate particular solutions of (69), each pair will furnish a single equation of relation as in previous cases, and assuming both to coexist for a twisted plane ray we find these equations of relation to be

[1] Trans. Roy. Irish Acad., Vol. XVII., 1836. See also Theory of Light, Thomas Preston, p. 377.

$$\begin{aligned} m^2 &= v^2(q-p)^2 + f(q-p)^3, \\ m^2 &= v^2(q+p)^2 - f(q+p)^3. \end{aligned} \tag{70}$$

By subtraction, etc., we have

$$\begin{aligned} f(q^2+3p^2) &= 2pv^2, \\ \therefore \frac{fq}{2v^2} &= \frac{p}{q}\left(1 - \frac{3p^2}{q^2} + \text{etc.}\right) = \frac{p}{q} \text{ nearly,} \end{aligned} \tag{71}$$

from which it is evident that f and p have essentially the same sign. By (70),

$$m = v(q-p)\left[1 + \frac{(q-p)}{2v^2}f - \frac{(q-p)^2}{8v^4}f^2 + \text{etc.}\right],$$

and (72)

$$m = v(q+p)\left[1 - \frac{(q+p)}{2v^2}f - \frac{(q+p)}{8v^4}f^2 - \text{etc.}\right].$$

Add equations (72) etc., then by (71) we have

$$v_0 = \frac{m}{q} = v\left(1 - \tfrac{3}{2}\frac{p^2}{q^2}t \text{ etc.}\right), \tag{73}$$

which in so far as the first two terms of the expansion are concerned is identical with (40). If (71) be substituted in (72) the approximate results will be found to be identical with (39) up to and including terms in p^2/q^2. Hence it appears that no experimental investigations as to the increase or decrease in the velocity of twisted plane polarized rays with apparatus now available will suffice to distinguish between the two electromagnetic hypotheses, to say nothing of distinguishing MacCullagh's elastic theory from these, which would require a refinement of still higher order.

III.

During the entire investigation up to this point the case of a perfectly transparent medium has been under consideration, *i. e.*, one which transmits light with undiminished amplitude, or at least with no absorption which needs be taken account of. It seems desirable however to investigate the theoretical effect of a small amount of absorption in order to show that no important modification of our results may thereby be introduced.

In case of absorption equations (25), according to Drude, take a somewhat more general form.

$$\begin{aligned}(1-ix)^2\frac{\partial^2 X}{\partial t^2} &= v^2\frac{\partial^2 X}{\partial z^2}+f\frac{\partial^3 Y}{\partial t\partial z^2},\\ (1-ix)^2\frac{\partial^2 Y}{\partial t^2} &= v^2\frac{\partial^2 Y}{\partial z^2}-f\frac{\partial^3 X}{\partial t\partial z^2},\end{aligned} \tag{74}$$

in which x is the so called coefficient of absorption, and the other symbols have the meanings previously given them in (25). Whether f, the small constant proportional to the strength of the field, is in this case complex or not seems to depend principally (as appears from the investigation of Drude previously referred to) upon whether the Amperian currents die out or not, *i. e.*, upon whether the energy of the orbital motions of the ions caused by the field suffers dissipation, or not. We shall assume that this kind of absorption or decay of energy if not entirely absent is at least very small, *i. e.*, if we take $f=f'+if''$, then f'' is negligibly small compared with f'. The consequences of this assumption will need to be considered to learn whether the results are self-consistent to a degree which will render it admissible.

Since the coefficients in (74) are not all real, it will be necessary to take the more general complex form (56) or its equivalent as the particular solution of (74) to represent the case of a twisted plane polarized ray, and to introduce the effect of absorption we must also put

$$R=Ae^{-rz}. \tag{75}$$

When this has been done the solution is of the form

$$\begin{aligned}X_1+X_2&=X,\\ Y_1+Y_2&=Y,\end{aligned} \tag{76}$$

in which if we write the results in exponential instead of trigonometrical form we have

$$X_1=\tfrac{1}{2}Ae^{i[mt-(q-p-ir)z]},\quad Y_1=-\frac{i}{2}Ae^{i[mt-(q-p-ir)z]}, \tag{77}$$

$$X_2=\tfrac{1}{2}Ae^{i[mt-(q+p-ir)z]},\quad Y_2=\frac{i}{2}Ae^{i[mt-(q+p-ir)z]}, \tag{78}$$

which equations differ from (57) and (58) when the latter are written in exponential form only in having $q - ir$ in place of q; they represent respectively the right- and left-handed circularly polarized rays in process of gradual extinction by absorption which when combined form the twisted plane polarized ray. Taking X_1 and Y_1 from equation (77) as particular solutions of both of equations (74) we find a single equation of relation for the right-handed ray, viz.,

$$(1 - ix_1)^2 m^2 = (q - p - ir)(v^2 + mf) \tag{79}$$

Assume f to be real, extract the square root and equate separately the reals and the imaginaries, then

$$m = (q - p)(v^2 + mf)^{\frac{1}{2}},$$

and

$$mx_1 = r(v^2 + mf)^{\frac{1}{2}}; \tag{80}$$

$$x_1 = \frac{r}{q - p}, \tag{81}$$

$$v_1 = \frac{m}{q - p} = v\left(1 + \frac{mf}{v^2}\right)^{\frac{1}{2}} \tag{82}$$

from which it appears by (27) that provided f be real the velocity v_1 is unaffected by the absorption. Similarly for the left-handed circularly polarized ray substitute the values of X_2 and Y_2 from (78) in (74), then we find from both the single equation

$$(1 - ix_2)^2 m^2 = (q + p - ir)^2 (v^2 - mf);$$

$$\therefore m = (q + p)(v_2 - mf)^{\frac{1}{2}},$$

$$mx_2 = r(v^2 - mf)^{\frac{1}{2}},$$

$$x_2 = \frac{r}{q + r}, \tag{83}$$

$$v_2 = \frac{m}{q + p} = v\left(1 - \frac{mf}{v_2}\right)^{\frac{1}{2}},$$

showing by (33) that the velocity is unaffected by the absorption.

Once more, if equation (76) of the twisted plane polarized ray be taken as a solution of (74) we have

$$m^2(1 - ix)^2[e^{(i\phi_1 - rz)} + e^{(i\phi_2 - rz)}]$$

$$= (q - p - ir)^2(v^2 + mf)e^{(i\phi_1 - rz)} \qquad (84)$$
$$+ (q + p - ir)^2(v^2 - mf)e^{(i\phi_2 - rz)}$$

Since the coefficients of each exponential in (84) must vanish separately we have

$$\begin{aligned} m^2(1 - i\varkappa)^2 &= (q - p - ir)^2(v^2 + mf), \\ m^2(1 - i\varkappa)^2 &= (q + p - ir)^2(v^2 - mf). \end{aligned} \qquad (85)$$

Still, assuming f to be real, extracting the square root and equating reals and imaginaries, we have

$$m = (q - p)(v^2 + mf)^{\frac{1}{2}}, \qquad (86)$$
$$m = (q + p)(v^2 - mf)^{\frac{1}{2}},$$
$$m\varkappa = rv(1 \pm mf/v^2)^{\frac{1}{2}}. \qquad (87)$$

The two values of $\varkappa$ shown by the contradictory signs in (87) are in disagreement because f has been assumed to be entirely real, when in fact it is complex, though the imaginary part of it is small. These two values of $\varkappa$ are brought into approximate agreement by writing in place of (87) the approximate equation,

$$m\varkappa = rv, \qquad (88)$$

in which it is to be specially noticed that its admissibility depends upon the smallness of f and not upon the size of $\varkappa$ or r; that is, the closeness of the approximation is independent of the amount of the absorption, but depends upon the amount of the twist. This approximation no doubt holds true within experimental limits.

Now, equation (88) may be written

$$\varkappa = \frac{r}{m}v = \frac{r}{q} \cdot \frac{v}{v_0} = \frac{r}{q} \text{ nearly}, \qquad (89)$$

which is a value of $\varkappa$ analogous to the values of $\varkappa_1$ and $\varkappa_2$ in (81) and (85), *i. e.*,

$$r = \varkappa_1(q - p) = \varkappa_2(q - p) = \varkappa q. \qquad (90)$$

Equations (86) are the same as (26) and (32) which were found in case of no absorption. Hence according to (74), based on the theory of orbital motion of ions, absorption leaves the phenomena previously investigated practically unchanged.

Again, let us consider whether the hypothesis of the Hall effect in an absorbing medium leads to the same approximate results. Drude's equations in this case may be written in the form

$$\begin{aligned} \frac{\partial^2 X}{\partial t^2} &= \frac{v^2}{(1 - ix)^2} \cdot \frac{\partial^2 X}{\partial z^2} + if \frac{\partial^2 Y}{\partial t^2}, \\ \frac{\partial^2 Y}{\partial t^2} &= \frac{v^2}{(1 - ix)^2} \cdot \frac{\partial^2 Y}{\partial z^2} - if \frac{\partial^2 X}{\partial t^2}, \end{aligned} \tag{91}$$

in which it will be assumed that the imaginary part of f is negligibly small. It will be noticed that by placing the complex factor in the denominator of one term, on clearing of fractions the coefficients of the two remaining terms of each equation become complex. The purpose of this artifice is to simplify the subsequent analysis.

Now take equations (77) as particular solutions of (91), we then find the single equation of relation

$$m^2(1 - ix)^2(1 - f) = v^2(q - p - ir)^2, \tag{92}$$

and assuming f to be real, we have after extracting the square root and equating reals, etc.,

$$\begin{aligned} &m(1 - f)^{\frac{1}{2}} = v(q - p), \\ &mx_1(1 - f)^{\frac{1}{2}} = vr, \\ &\therefore x_1 = r/(q - p), \end{aligned} \tag{93}$$

for the right-handed circularly polarized ray. Similarly taking equation (78) we have from (91)

$$m^2(1 - ix)(1 + f) = v^2(q + p - ir)^2, \tag{94}$$

$$m(1 + f)^{\frac{1}{2}} = v(q + p),$$

$$mx_2(1 + f)^{\frac{1}{2}} = vr, \tag{95}$$

$$x_2 = r/(q + p),$$

for the left-handed ray.

Finally taking (76) as a particular solution of (91), by equating coefficients of the separate exponentials we have equations (92) and (94) simultaneously true, which may also be obtained from equations (59) and (60) by replacing m and q, by $m(1 - ix)$ and $(q - ir)$ respectively.

Hence if f be assumed to be real we have

$$\begin{aligned} m(1 \mp f)^{\frac{1}{2}} &= v(q \mp p), \\ mx(1 \mp f)^{\frac{1}{2}} &= vr. \end{aligned} \tag{96}$$

The disagreement between the values of x shown by the ambiguous signs in the last equation is to be overlooked by placing

$$mx = vr, \tag{97}$$

which equation is practically accurate within experimental limits by reason of the smallness of f. From this we see that equations (89) and (90) hold in this case also.

The first of equations (96) is identical with equations (61) and (62) and consequently the same as in case of no absorption.

SUMMARY.

The conclusions arrived at in this paper may be briefly stated as follows:

1. The increase or decrease in the velocity of circularly polarized light observed by Mr. Mills, and previously by Professor Brace, is perfectly in accord with and a necessary consequence of the elementary trigonometrical equations expressing the propagation of twisted plane polarized light, and the phenomenon is independent of any hypothesis electromagnetic or otherwise as to the manner in which the twisting is produced.

2. The equations given by Professor Rowland to express the propagation of twisted plane polarized light are not suitable for that purpose, for they in fact express the propagation of a uniformly and a continuously rotated plane polarized ray, such as is at present unknown to experimental physics.

3. The velocity of a twisted plane polarized ray is so related to the velocities of the right and left circularly polarized rays of which it is composed that its reciprocal is the arithmetical mean of the reciprocals of its components; and the velocity of Professor Rowland's rotating plane ray is the arithmetical mean of the velocities of its right and left circularly polarized components.

4. The differential equations based on two different electromagnetic hypotheses as to the action of the medium in producing rota-

tion or twisting of plane polarized light in a magnetic field involve an equation expressing the relation of the velocity of this kind of light at zero field to its velocity in the given field. On the hypothesis of orbital motions of charged ions the differential equations show that the field would cause a decrease in the velocity of plane polarized light during the twisting; while on the hypothesis of charged ions having a motion of translation across the field, the differential equations show that the field would cause an increase in the velocity by an amount one third as great as the decrease just mentioned. This increase or decrease is of the second order of small quantities, and is so minute as to be at present beyond the range of observation, varying as it does as the square of the observed change produced by the field in the velocity of circularly polarized rays.

5. MacCullagh's differential equations involve practically the same decrease of velocity by the medium as those based on orbital motions of charged ions.

6. A moderate amount of absorption in the medium would not practically modify the conclusions true for perfectly transparent media on either hypothesis.

EXPERIMENTS ON RESONANCE IN WIRELESS TELEGRAPH CIRCUITS. PART II.

BY GEORGE W. PIERCE.

THIS paper is a continuation of a research[1] published under the same title in the PHYSICAL REVIEW, Vol. XIX, No. 3, September, 1904. The figures, sections, and tables are numbered continuously with those of the previous paper.

The circuits employed are again of the electromagnetically connected type represented in Fig. 10. It is seen that the system is constituted of four circuits, which shall be referred to in what follows as I., the sending side circuit; II., the sending mast circuit; III., the receiving mast circuit, and IV., the receiving side circuit.

The experiments here described treat of variations of the length and form of the antennæ, and the resulting effect on resonance. It is shown that the proper adjustment of the antennæ is of primary importance. The proper adjustment is not attained by making the mast circuits at the two stations *equal*. Even with the mast circuits at the two stations constituted of coils and antennæ of exactly the same dimensions, it is possible, under conditions that are not unlikely to occur, to obtain the singular result that the amount of energy in the side circuit at the receiving station is insignificantly small, while at the same time the capacity of the condenser in this circuit must be made infinite to obtain resonance.

The present experiments also show that the resonance conditions are multiple-valued, so that unless the circuits are well understood there is danger of setting the receiving circuit to resonance with some feeble oscillation and thus failing to take advantage of a more energetic train of waves.

[1] The proofs of the previous paper were read in the absence of the proofs of the figures, and a number of errors were made in the references to curves. These errors are here corrected:

Page 207, line 2, instead of "16" read "17"; page 210, line 1, instead of "microfarads" read "farads"; line 4, instead of "17 and 14" read "18 and 15"; last line instead of "14" read "15"; page 211, line 19, instead of "14" read "15"; page 214, line 12, instead of "17" read "15."

The fact, however, that the conditions for resonance are multiple-valued does not preclude the possibility of simultaneously operating a large number of wireless telegraph stations without interference, because under proper conditions the resonance is sharp, and the number of possible variations of the circuits is large.

The results of the experiments; in particular, the formulas obtained for the relation between the capacity of the antenna and the

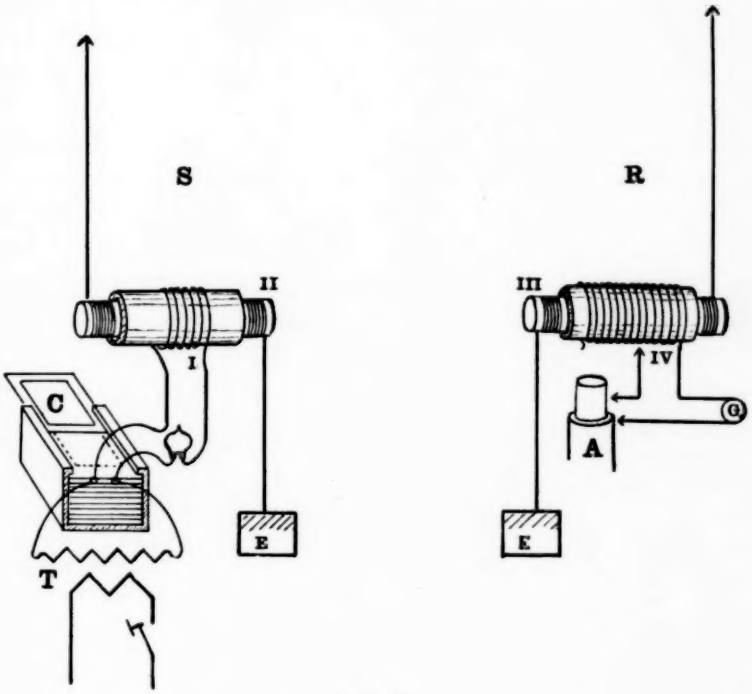

Fig. 10.

resonant capacity in the side circuit, are found to agree with deductions from the elementary theory of alternating currents.

V. Apparatus.

The apparatus used in these experiments is essentially the same as that of the previous research, except that the stations were set up farther apart, and provisions were made for longer antennæ, better ground connections, and for better adjustment of the range of the receiving instrument.

The distance between the two stations is 187 meters across an open field.

Antennæ 25 meters high could be used. They were carefully insulated from their supports and from the roofs of the station houses.

Ground. — The ground at each station consisted of three copper plates and a piece of iron pipe. The copper plates, 60 cm. square by .3 cm. thick, were embedded in the earth 270 cm. below the surface. The pipe, 3 cm. in diameter and 180 cm. long, was driven in until flush with the bottom of the hole containing the plates, and, therefore, reached to a depth of 450 cm. below the surface. The single pipe proved to be a much better ground than the three pieces of copper; in fact, measurements of current in the mast circuit at the two stations indicate that the plates added very little to the pipe, showing that in this particular soil depth counted for more than did extent of surface in securing a good contact with the earth.

Coils. — Throughout these experiments the coils of the four circuits were kept constant and had the following dimensions:

Coil No.	No. of Turns.	Diameter of Wire, cm.	Length of Solenoid, cm.	Inductance, Henries.
I.	9	.164		1.71×10^{-5}
II.	240	.104	46	125×10^{-5}
III.	240	.104	46	125×10^{-5}
IV.[1]	17	.164		7.04×10^{-5}

[1] Including the instrument.

Condensers. — The condensers at the two stations were adjustable. The condenser at the sending station was made of copper foil separated by plates of glass. The number of plates could be varied, and smaller changes could be obtained by sliding the top plate. At the receiving station air condensers were used. They were four in number, made of concentric tubes of brass. The capacity in this circuit was varied by throwing in or out these air condensers as wholes or by varying any one of them by withdrawing the inner cylinder and reading on a paper scale the number of centimeters of length left overlapping. In the curves presented in this paper, the capacity in the receiving side circuit, called "receiving capacity"

is given in cm. of cylinder overlapping in the air condensers — 1 cm. being equal to 2.77×10^{-11} farads.

Mercury Interrupter. — The oscillations at the sending station were produced by the discharge of the glass condenser through a Cooper Hewitt Mercury Interrupter.[1] The mercury interrupter was submerged in oil kept at 95° by an electric heater controlled by an automatic thermal regulator. It was shown in the previous part of this research[2] that a mercury interrupter, in which no residual air was left, operated most effectively at that temperature. In the present experiments, in addition to heating the mercury interrupter, it was continually shaken by a small electric motor so as to keep always a fresh surface of mercury and to prevent deterioration as a result of the discharge taking place too long from a single point in the neighborhood of the glass wall of the bulb. It was found that this continual shaking of the mercury interrupter kept externally at a constant temperature served to make it as nearly constant in its action as could be desired. To show the regularity thus attainable Table V. gives seventy readings of a Kendrick hot-wire ammeter placed between the coil and the ground in the mast circuit at the sending station.

TABLE V.

Showing Regularity of the Mercury Interrupter when Heated to a Constant Temperature and Continually Shaken.

9.5	9.3	9.4	9.4	9.6	9.5	9.65
9.4	9.3	9.4	9.4	9.5	9.6	9.6
9.3	9.5	9.5	9.4	9.4	9.5	9.4
9.4	9.4	9.3	9.4	9.5	9.6	9.6
9.5	9.4	9.4	9.6	9.4	9.6	9.5
9.3	9.4	9.4	9.5	9.5	9.5	9.5
9.3	9.4	9.4	9.6	9.6	9.65	9.6
9.3	9.4	9.3	9.6	9.5	9.5	9.65
9.4	9.4	9.55	9.6	9.4	9.5	9.65
9.3	9.55	9.4	9.3	9.6	9.4	9.4

To reduce these readings to amperes, divide by 5.

The source of current in these experiments was a step-up transformer operated on the 110-volt alternating electric light circuit. The secondary of the transformer was connected to the condenser

[1] Pierce, Proc. Am. Acad., XXXIX., No. 18, Feb., 1904.
[2] Pierce, PHYSICAL REVIEW, XIX., p. 216.

C, Fig. 10. The switch in the primary was closed and opened automatically by a clock-work, so that the signals were sent every 35 seconds, without the aid of an assistant. Each signal lasted for 5 seconds, which was a little greater than the time required for reading the receiving instrument.

Receiving Instrument. — One form of the receiving instrument, which might be called a "high frequency dynamometer" is described in the previous paper. It consists of a small coil (8 or 9 mm. in diameter) connected in series with the condenser circuit of the receiving station. Immediately in front of the coil is hung a plane glass mirror (3 mm. in diameter) backed up by a thin disc of silver of approximately the same diameter. The disc makes an angle of 45° with the plane of the coil. Oscillations in the coil induce oscillations in the disc, which is deflected so as to increase the angle between the disc and the coil. The deflections are read by means of a telescope and scale, with scale distance of 90 cm.

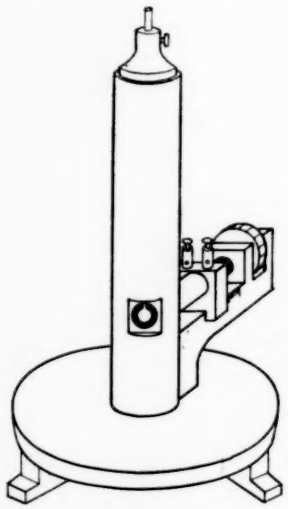

Fig. 11.

In the present experiments the coil of the instrument had 30.5 turns of wire .11 mm. in diameter, and by moving the coil toward or away from the suspended disc, the sensitiveness of the instrument could be readily changed without changing the resistance or appreciably changing the inductance of the receiving circuit. The instrument thus has a very wide range, and was used in one of the experiments here described to compare directly quantities of energy that were in the ratio of 1 to 20,000.

The motion of the coil toward or away from the suspended disc was effected in the present experiments by means of a micrometer mounting shown in Fig. 11.

VI. Theory and Calibration of the Instrument.

The rubber screw of the micrometer used in moving the coil of the instrument toward the suspended disc has a pitch of $\frac{1}{32}$ of an

inch, and the scale of the micrometer reads in thirty seconds of an inch with an arbitrary zero. The head of the screw is divided in tenths. To enable the use of the instrument to measure widely different quantities of energy, the relative sensitiveness of the instrument for various positions of the coil was determined by the following experiment.

Experiment IV. Calibration of the Micrometer Scale of the High Frequency Dynamometer. — With a constant source of waves and a constant receiving circuit, deflections were taken with the coil of the instrument at various positions, that is, with various readings of the micrometer by which the coil is moved. The following sets of readings, Table VI., were obtained.

TABLE VI.

Calibration of Sensitiveness Scale on Instrument.

Micrometer.	7.5	8.0	9	10	11	12	12.5	13
Deflection in cm. observed.	2.23	2.87	4.87	9.0	16.2	29.7	39.0	52
	2.27	2.87	4.87	8.9	16.4	30.2	38.2	
	2.17	2.82	4.83	8.6	17.0	30.7	38.0	
	2.27	2.82	5.00	8.8	16.5	29.0	38.5	
Mean observed.	2.23	2.85	4.89	8.8	16.5	29.9	38.4	52
Calculated.	2.15	2.85	5.00	9.0	16.2	29.5	38.6	51

Micrometer.	3	5	7
D. calculated.	.28	.67	1.70

The average observed values of Table VI. are plotted in the curve of Fig. 12. This curve was used for the comparison of quantities of energy that differed so widely that it was necessary to use different sensitivenesses of the instrument in taking their readings.

The Calculated Values. — The observed values in Table VI. agree satisfactorily with the values in the last row of the table which were calculated by the formula

$$\frac{D_1}{D_2} = \left\{ \frac{(16 - m_2)^2 + 25}{(16 - m_1)^2 + 25} \right\}^3 \qquad (1)$$

in which m is the reading of the micrometer of the coil, and D the corresponding deflection.

The formula (1) is a rational equation obtained on page 9 in the discussion of the theory of the dynamometer. If the reader wishes to omit the discussion of the theory, he will find the conclusions in regard to the instrument stated in equation (12) and in the succeeding paragraphs on page 13.

The proposition (2) page 13 that the deflections are proportional to the sixth power of the distance from the center of the movable disc to the periphery of the coil gives equation (1), in which the $16 - m$ is the distance from the center of the coil to the center of the disc, and 25 is the square of the radius of the coil with the pitch of the screw as unit. Thus $\sqrt{(16-m)^2+25}$ is the quantity C in equation (12).

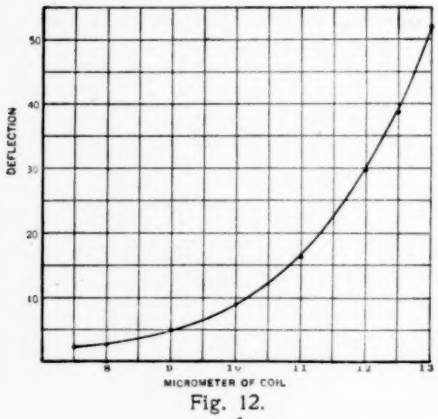

Fig. 12.

The agreement between the observed values of Table VI. and values calculated from equation (1) is so close that I have used equation (1) for determining the relative sensitiveness of the instrument for values of m between 3 and 7, where direct measurements were not made. These values are also given in Table VI.

Digression on the Theory of the Dynamometer. — To simplify the reasoning let us assume that the coil of the instrument consists of n turns of fine wire so packed together that the thickness of the coil may be neglected in comparison with its radius, and that the movable system is a ring consisting of a single turn of fine wire.

If φ is the angle between the axis of the coil and the axis of the ring, the moment of the force tending to increase φ is

$$F_m = i_4 i_5 \frac{dM}{d\varphi}, \qquad (2)$$

in which i_4 and i_5 are respectively the current in the coil and the ring, and M the mutual inductance between the coil and the ring. Let us get approximate values for these three quantities. Assuming he current in the coil sinusoidal,

$$i_4 = I_4 \sin \omega t. \tag{3}$$

The e.m.f. induced in the ring is

$$e_5 = \frac{d}{dt}(-i_4 M)$$

$$= -M\frac{di_4}{dt} - i_4\frac{dM}{dt}.$$

For small deflections $\frac{dM}{dt}$ may be put equal to zero; then

$$e_5 = -MI_4\omega \cos \omega t.$$

The current in the movable ring must satisfy the ordinary e.m.f. equation

$$L\frac{di_5}{dt} + Ri_5 = -MI_4\omega \cos \omega t,$$

in which L and R are the self-inductance and resistance of the movable ring. The solution of this equation is, for the steady state,

$$i_5 = \frac{-MI_4\omega \cos(\omega t - \alpha)}{(R^2 + L^2\omega^2)^{\frac{1}{2}}}; \tag{4}$$

$$\alpha = \tan^{-1}\frac{L\omega}{R}. \tag{5}$$

Whence from equations (3), (4) and (2),

$$F_m = \frac{-MI_4^2\omega \sin \omega t \cos(\omega t - \alpha)}{(R^2 + L^2\omega^2)^{\frac{1}{2}}} \tag{6}$$

The average value of this deflecting moment is

$$F_m = \frac{1}{T}\int_0^T F_m dt = -\frac{LM\omega^2 I_4^2}{2(R^2 + L^2\omega^2)}\cdot\frac{dM}{d\varphi}. \tag{7}$$

This is an expression for the deflecting moment in terms of the current in the coil, the resistance and self inductance of the ring and the mutual inductance between the coil and the ring. Now the mutual inductance between two circular circuits whose planes make an angle φ with each other is given by the series

$$M_1 = \frac{1 \cdot 2\pi^2 A^2 a^2 \cos\varphi}{C^3} + \frac{2 \cdot 3 \cdot \pi^2 A^2 B}{C^5} a^2 b(\cos^2\varphi - \tfrac{1}{2}\sin^2\varphi)$$

$$+ \frac{3 \cdot 4\pi^2 A^2 (B^2 - \tfrac{1}{4}A^2)}{C^7} \cdot a^2(b^2 - \tfrac{1}{4}a^2)(\cos^3\varphi - \tfrac{3}{2}\sin^2\varphi\cos\varphi) \quad (8)$$

$$+ \text{etc.} \ldots^1$$

This is the expression for the mutual inductance between the movable ring and *one turn of the coil*, with the following values for the constants. A and a are the radii of the coil and ring respectively, B and b the distance of their planes from the origin, and

$$C = \sqrt{A^2 + B^2}.$$

Choosing the origin at the center of the movable ring, b becomes zero, causing the second term of M_1 to disappear. For the dimensions of coil and ring in the instrument (A = about .4 cm. and B greater than .1 cm.) the third term may also be neglected with an error not greater than 3 per cent.

$$\therefore M_1 = \frac{2\pi^2 A^2 a^2 \cos\varphi}{C^3} \pm 3 \text{ per cent.} \quad (9)$$

$$\text{For } \varphi = 45°, \frac{dM_1}{d\varphi} = -M_1, \quad (10)$$

For n turns in the coil,

$$M = nM_1. \quad (11)$$

Substituting (9), (10) and (11) in (7) and replacing $\cos\varphi$ by $1/\sqrt{2}$ we obtain

$$F_m = \frac{Ln^2\omega^2 I_4^2}{R^2 + L^2\omega^2} \cdot \frac{\pi^4 A^4 a^4}{C^6}. \quad (12)$$

In this equation R and L are the resistance and self-inductance of the movable ring, n the number of turns on the coil, ω the angular velocity of the oscillation in the coil, A and a the radii of the coil and ring respectively, and I_4 the current in the coil.

Conclusions from the Theory of the Dynamometer. — Since the deflections are proportional to the deflecting moment, we may say that the deflections are: (1) Directly proportional to the square of the current in the coil, (2) inversely proportional to the sixth power of the distance from the center of the movable disc to the periphery

[1] Maxwell, El. and Mag.

of the coil, (3) proportional to the square of the number of turns in the coil, and (4) generally involving the frequency in a manner dependent on the relative size of R^2 and $L^2\omega^2$. If $L^2\omega^2$ is negligible in comparison with R^2, the deflections are proportional to the square of the frequency; while, on the contrary, if R^2 is negligible in comparison with $L^2\omega^2$, the deflections are independent of the frequency of the incident waves.[1]

The agreement between the observed and calculated values of the deflections shown in Table VI., Experiment IV., is proof of the second of the propositions just stated. To test the first of the propositions, the following experiment was performed.

Experiment V. — Showing that the Deflections are Proportional to the Square of the Current in the Coil. — With the sending and receiving circuits in resonance, deflections were taken with various values of a known resistance put in series with the coil of the instrument. The known resistance consisted of two parallel manganine wires of which the length could be varied by a sliding contact across the two wires. Changing the position of the slide did not appreciably change the inductance of the circuit.

Since the circuits were in resonance, their impedance was wholly in the nature of a resistance, and if the deflections are proportional to the square of the current, we should have,

$$D = \frac{k^2}{(R + x)^2},$$

in which R is the added resistance and k and x are constant. Using the case $R = 0$ to determine $k(= 85.4)$ the deflections and the calculated values of x obtained for various values of R are given in Table VII.

TABLE VII.

Showing Deflections Proportional to I_4^2.

R Added.	Deflection cm.	x Calc.	R Added.	Deflection cm.	x Calc.
0	41.8	13.4	8.10	15.8	13.4
2.74	27.3	13.8	10.9	12.3	13.5
5.63	20.4	13.3	18.7	7.1	13.3

[1] It should be remembered that when the instrument is used to measure induced currents, the frequency also enters in the factor I_4.

While this is not a direct proof of the proposition, the approximate constancy of x is held in support of the theoretical deduction that the deflections are proportional to the square of the current in the instrument.

VII. Resonance Curves with Variation of the Length of Antennæ.

The purpose of the experiments that follow is to investigate the effect on resonance, produced by varying the form and height of the antennæ.

Experiment VI. Four Wires Sending and Four Wires Receiving. —A sending antenna was chosen with the form and dimensions shown in Fig. 13. The four nearly vertical wires of copper are .208 cm. in diameter and run parallel for a distance of 11.6 meters. They are braced apart at a distance of .59 meters from each other by two cross pieces of *brass tubing* .8 cm. in diameter. The metal

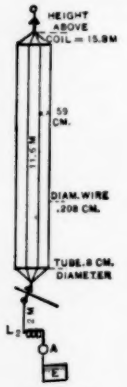

Fig. 13.

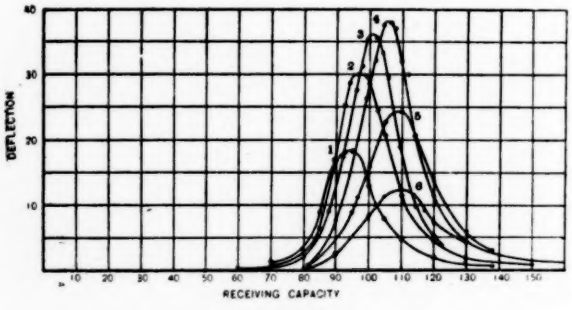

Fig. 14.

was chosen for the cross pieces instead of an insulating material, because its conductivity and the resulting capacity of the conductor would not change with the accumulation of moisture.

The total height of the sending antenna (Fig. 13) measured from the coil is 15.8 meters.

After choosing the sending antenna, the next step was to choose the proper capacity in the side circuit at the sending station; that is, to bring the side circuit at the sending station to resonance with the mast circuit. The process was as follows: With the receiving

mast circuit of the same dimensions as the sending mast circuit, resonance curves, as in the previous paper, were taken for several values of the capacity in the sending side circuit. These curves are shown in Fig. 14.

The different curves correspond to different values of the condenser at the sending station, and the individual points of each curve are the deflection obtained with various values of capacity of the air condenser in the side circuit at the receiving station. The details of the method of obtaining these curves are given in the previous paper.

Curve 4 shows the best resonance, and the capacity in the sending side circuit was set at the value that gave curve 4 and kept at that value throughout Experiment VI.

TABLE VIII.

Capacities Giving the Curves of Fig. 14.

Curve No.	Sending Capacity Microfarads *S.*	Resonant Receiving Capacity Microfarads *R.*	Ratio *S*/*R.*
1	127×10^{-4}	25.5×10^{-4}	4.98
2	139	26.5	5.25
3	151	28.0	5.40
4	163	29.1	5.61
5	176	29.9	5.79
6	189	30.4	6.23

With the two circuits at the sending station thus in unison with each other the resonance conditions at the receiving station were studied with various lengths of receiving antenna.

We have then coming to the receiving station waves of constant period and approximately constant intensity, and we are to seek, for several lengths of receiving antenna, the conditions under which the receiving instrument shows the largest readings. The variables are now the height of the receiving antenna and the capacity of the air condensers, which are in the side circuit at the receiving station.

The receiving antenna of four wires was started at a height of 23.8 meters, measured from the coil in the mast circuit. The cross pieces and converging portions were of the same dimensions as at the sending station. The receiving antenna in this case was eight meters higher than the sending antenna. With this arrangement

Curve 1, Fig. 15, was taken by observing the deflections when the air condenser in the receiving side circuit was set at various values

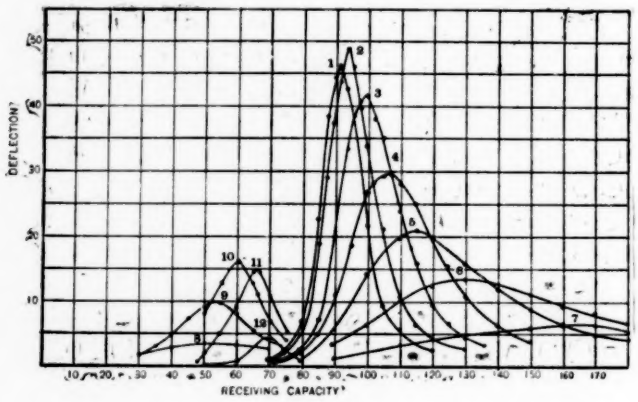

Fig. 15.

from 70 to 180 cm. of overlapping cylinder, each cm. of overlapping condenser having a capacity of 2.77×10^{-11} farads.

Next the receiving antenna was shortened by cutting off 3 meters from the parallel portion, making the height 20.8 meters. Curve 2, Fig. 15, was obtained. Decreasing the length further to 17.8, 15.8, 14.8, 13.8 and 12.8 meters gave Curves 3, 4, 5, 6 and 7 respectively of Fig. 15. When the antenna was decreased one half meter further, the deflections were smaller than those of Curve 7, and increased slowly out to the limit of my available receiving capacity (180 cm. cyl.), so that the maximum could not be located.

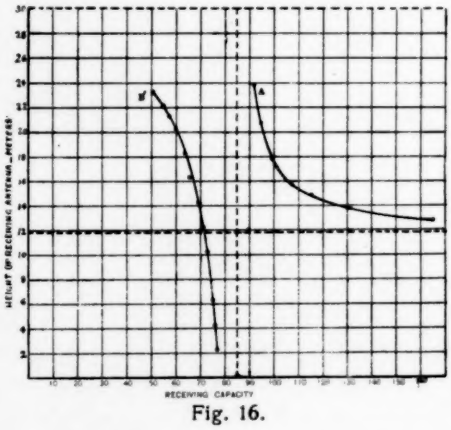

Fig. 16.

Full discussion of Curves 1 to 7 will be taken up later. We shall here note only an empirical relation between the height of the antenna and the capacity in the side circuit at the receiving station.

Empirical Equation for the Relation of H_a to C_4. — The above

curves show that with a fixed frequency of incident waves, when the height of the receiving antenna H_a was *decreased*, it was necessary to *increase* the condenser capacity C_4 in order to obtain resonance.

To show quantitatively this effect Curve *A*, Fig. 16, was constructed with the resonant capacity in the receiving side circuit as abscissa and the height of the antenna as ordinate.

Curve *A*, Fig. 16, has approximately the empirical equation

$$(H_a - 11.8)(C_4 - 84.6) = 88, \qquad (a)$$

as shown by the following comparison of observed with calculated values, Table IX.:

TABLE IX.

Relation Between Height of Receiving Antenna and Resonant Capacity. Four Wires Receiving.

Curve No., Fig. 15.	Meters Antenna Above Coil, H_a	Maximum Deflection, cm.	Resonant Capacity, Observed, C_4	Resonant Capacity Calculated.
1	23.8	46	92	91.9
2	20.8	47	94	94.4
3	17.8	43	100	99.3
4	15.8	29.5	106	106.6
5	14.8	21	115	113.9
6	13.8	13	130	128.6
7	12.8	7.5	165	172.6

The only large difference between the observed and the calculated value of resonant capacity is in the case of Curve 7, where on account of the obtuseness of the experimental curve, its maximum could not be accurately determined.

Experiment VII. Four Wires Sending and One Wire Receiving. — The sending station was left as it was in the preceding experiment. To see whether decreasing the number of wires in the receiving antennæ had the same effect as cutting off wire from the four wires, a receiving antenna of various lengths of a *single* wire was employed, and the corresponding resonance curves were taken as in Experiment VI. The individual curves are not presented, but the positions of their maxima are shown in Curve *B'*, plotted in Fig. 16, with resonant capacity of the air condenser as abscissæ and the height of antenna as ordinates. In going from the four wires to one wire the resonant capacity C_4 had to be reduced, but had to be increased

again as length of wire was cut off. Thus from Experiments VI. and VII. we have: (1) Cutting off from the four-wire antenna increases C_4; (2) changing from four wires to one wire decreases C_4; (3) cutting off from the one-wire antenna increases C_4; (4) the curve B' with one wire curves oppositely from the curve A with four wires.

The apparent contradictions existing among these propositions disappear when we discover that the equation to curve B' has the same form as the equation to the curve A. The approximate equation for curve B' is

$$(H_b - 30)(C_4 - 84.6) = 230, \qquad (b)$$

as shown by the comparison of observed and calculated values in Table X.

TABLE X.

Relation Between Height of Receiving Antenna and Resonant Capacity. One Wire Receiving.

Meters Antenna Above Coil H_b.	Maximum Deflection cm.	Resonant Capacity Observed, C_4.	Resonant Capacity Calculated.
23.3	6.7	50	50.2
22.3	9.3	55	54.6
21.3	13.1	57	58.2
20.3	14.5	60	60.9
18.3	13.6	63	65
16.3	8.0	65	67.8
14.3	5.2	69	70
12.3	3.0	71	71.6
10.3	1.3	72.5	72.9
8.3	.54	74	74
6.3	.16	75	74.9
4.3	.057	76.5	75.7
2.3	.016	77	76.3

The small values of the deflections in the second column were observed as large deflection with increased sensitiveness of the instrument and reduced. They are as accurate as the larger numbers in the same column.

Equation (b) has one constant (84.6) in common with equation (a). Curves A and B' are thus seen to be hyperbolas with the vertical asymptote in common. In what follows we shall see that the horizontal asymptote of B' would probably transform into that of A, and that B' would probably become the other branch of the hyperbola A, if instead of the height of the dissimilar antennæ we should plot the *capacity* of the antennæ as ordinates.

Instead of attempting to make this transformation, which would necessitate the measurement of the capacities of the antennæ, it was decided to seek the other branch of Curve *A* by going back to the antenna with four wires and continuing to decrease the height of the four-wire antenna below the value where the resonant capacity became too large for determination.

Experiment VIII. Four Wires Sending and Four Wires Receiving. Search for the Other Branch of the Hyperbola. — As the height of the four wires of the receiving antenna was decreased by small intervals below the values that gave Curve 7, Fig. 15, the deflections in the region of capacity between 90 and 180 became smaller and smaller, as if the resonant point were going away to infinity, and the deflections in the neighborhood of 50 began to grow, until when the height of the antenna was made 10.5 meters, a maximum became evident for about 50 cm. of the receiving condenser. The readings by which this maximum was obtained are plotted as Curve 8 in Fig. 15, along with Curves 1 to 7, which were also taken with four wires receiving. Decreasing the height still further, Curves 9, 10, 11 and 12 were obtained with respectively 10, 9, 8 and 7 meters as the height of the antenna. These curves increase in intensity up to Curve 10 and fall off in 11 and 12. The five curves 8 to 12 were taken with sensitiveness of the receiving instrument about five times as great as the sensitiveness used in taking Curves 1 to 7. Curves 8 to 12 in the left-hand group of Fig. 15 are plotted thus magnified five times in comparison with the group to the right, numbered 1 to 7.

The two groups when plotted with resonant receiving capacity against height of antenna form the two branches of the hyperbola *A*, *A'*, Fig. 17. The equation of the hyperbola is the equation (*a*) on page 19. Values calculated from this equation are plotted as the dotted lines in Fig. 17. The heavy curves are the observed values. We shall have occasion to discuss the imperfectness of the agreement, when we came to the discussion of the theory of the hyperbola relation.

The curves of Figs. 15, 16 and 17 may be discussed from three points of view: (1) With respect to the relation of capacities at resonance; (2) with respect to amount of energy receivable; (3) with respect to sharpness of resonance.

1. *With Respect to the Relation of the Capacities at Resonance*, it is seen that the height of the antenna and the resonant capacity of the condenser in the side circuit at the receiving station are connected approximately by the equation to a hyperbola with asymptotes displaced. As shown by the above curves and as will be shown by further curves the position of the horizontal asymptote depends, among other things, upon the nature of the antenna. With any form of antenna a fixed constant is substracted from the height to form the first factor of the equation.

This constant is a height of antenna. In the case of the heterogeneous antenna of four wires this height to be subtracted is 11.8

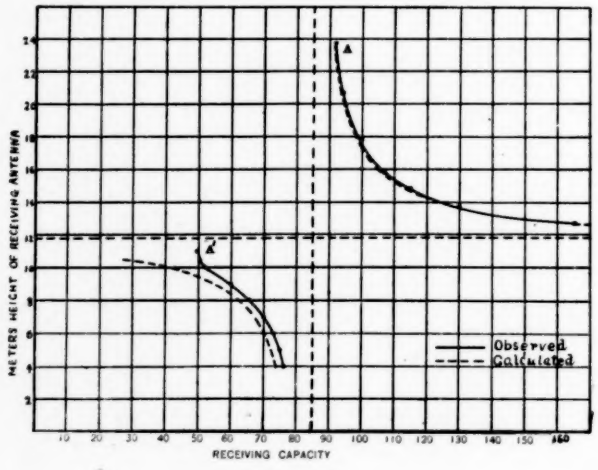

Fig. 17.

meters of antenna including rigging at top and bottom. When the constant height is subtracted from the variable height H_a, including also rigging at top and bottom, the difference is a length of *four parallel wires*, not complicated by cross pieces and converging portions. It is this fact that leads to the simplicity of equations (*a*) and (*b*), for even when a heterogeneous antenna is used in the experiments, the quantity that enters in the equation is a length of homogeneous conductor — which has a capacity approximately proportional to its length.[1]

[1] Compare Drude, Ann. d. Phys., 11, p. 957.

The significance of the hyperbola relation between the capacity of the mast and the capacity of the side circuit, in particular the meaning of the asymptotes is examined later, along with the theoretical discussion.

The curves of Fig. 15 and Fig. 17 were all taken with incident waves of the same frequency. The curves show that whatever the frequency of the incoming waves, and whatever the height of the receiving antenna, it is possible to get a resonant value (may be of small intensity) in an adjustable condenser circuit of sufficient range of capacity, unless the receiving antenna happens to have a height that would bring it in the neighborhood of the horizontal asymptote of the hyperbola. The position of this asymptote is later shown theoretically to be a function of the frequency of the incident waves and of the self and mutual inductances of the two parts of the receiving circuit.

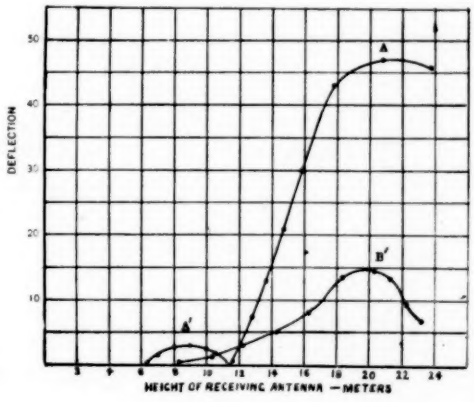

Fig. 18.

If the height of the receiving antenna happens to coincide with the vertical displacement of the horizontal asymptote, no resonance is possible between the two parts of the receiving circuit, and the energy receivable is insignificant. We shall see that this case might arise even when the sending and receiving mast circuits are of identical dimensions.

It is thus not correct to set up identical mast circuits at the two stations and tune by the adjustment of the side circuits.

We shall see later that, except in the special case in which the two stations are of the same dimensions in all their parts, the sending and receiving antennæ should have capacities different from each other or else the coils in their mast circuits should be different from each other.

2. *With Respect to the Amount of Energy Receivable.* — In Fig. 18 the deflections obtained in resonant side circuits (the maxima

of the curves of Fig. 15) are plotted against height of antennæ. These deflections, which are proportional to the square of the current in the instrument, are all on the same scale, and were verified by subsequent experiments in which runs were made merely of the maxima. The curve AA' is with the four-wire antenna (Experiments VI. and VIII.); B' is with the one-wire antenna (Experiment VII.). With the four wires a maximum is found at a height of 21 meters and another at 9 meters. These two maxima belong to heights in different branches of the hyperbola, Fig. 17. B' has a maximum at about 19.6 meters of height of the single wire. AA' has a zero value in the neighorhood of the asymptote, 11.8 meters.

The position of the maxima of A, A' and B' serve to test certain propositions of the theory of the circuits to be given later. For this purpose it is better to plot the maximum deflections against the receiving capacity resonant with each height of antenna, instead of against the height itself. When this is done, Fig. 19 (in which A'

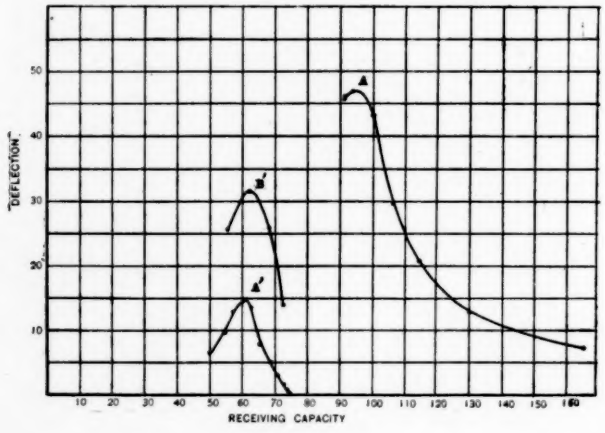

Fig. 19.

and B' are magnified five times) it is seen that the maximum of B' falls just above A'. The resonant capacities in the side circuit are the same in both cases. This leads to the inference that the capacities of the antennæ (19.6 meters of single wire and 9 meters of the heterogeneous four-wire mast) are the same; also that the length of single wire below the asymptotic value (30–19.6) has the same capacity as the length of the four wires below the corresponding

asymptotic value (11.8–9). These difference lengths are length of single wire in the first case, and length of four parallel straight wires in the second case; whence we may infer that the capacity of 10.4 m. single wire is equal to the capacity of 2.8 m. of four wires. These numbers are in the ratio of 3.7 to 1. If the four wires were far enough apart to be independent, this ratio should be 4 to 1. This is of course a very inaccurate way of comparing capacities, and the agreement is close enough to support the view that the capacity of the four wires and that of the single wire are the same when the resonant side capacities are the same.

On account of the prominence of resonance in determining the amount of energy received it is difficult to draw inferences as to the *effect of the height in simply affording a longer circuit for the waves to cut.* Curves A' and B', however, enable us to compare deflections (I^2) with different length of antennæ having the same capacity; that is, having the same resonant capacity in the condenser circuit. The following table, Table XI., contains the height of four wires and of one wire that have the same capacity. The greater deflection in the latter case is due simply to greater height without better resonance.

TABLE XI.

Comparison of Reflections for Different Heights of Antennæ that are Equally Resonant, as Determined by Common C_4.

C_4 Resonant Capacity.	H_4 Height of Four Wires.	H_1 Height of One Wire.	D_4 Deflection with H_4.	D_1 Deflection with H_1.	$\sqrt{\frac{D_1}{D_4}}$	$\frac{H_1}{H_4}$
50	10.4	23.2	1.5	6.7	2.11	2.33
60	9	19.6	3.0	14.7	2.21	2.18
70	7	13.2	1.3	4.0	1.75	1.89

Table XI. contains too few data and the readings of deflections happen to be too small and taken at too long intervals of time apart to justify the requirement of exactness, but the approximate agreement of ratios *in the same line* in the sixth and seventh columns indicate that the *current* (square root of the deflections) in the antenna is *proportional to its height.* A better test of this law is reserved for a later experiment.

3. *With Respect to Sharpness of Resonance* of the several curves of Fig. 15, Table XII. was compiled from the curves. The table

gives the percentage change in capacity required to cause the deflection to fall to half on that slope of the curve that is steepest.

TABLE XII.

Sharpness of Resonance.

Curve No.	Deflections Compared.	Percentage Change of Capacity.	Curve No.	Deflections Compared.	Percentage Change of Capacity.
1	30–15	3.2	7	5–2.5	23
2	30–15	3.6	8	3.5–1.75	40
3	30–15	5	9	10–5	20
4	20–10	6.2	10	16–8	17
5	20–10	13.6	11	15–7.5	12
6	12– 6	16.3	12	4–2	7.4

Experiment IX. Four Wires Sending and Eight Wires Receiving. — With the sending station the same as in Experiments VI., VII., and VIII., a receiving antenna of eight wires was used. The wires had the same diameter as those of the previous experiments. They were strung up between two metal hoops 118 cm. in diameter. The parallel portion of the wires was four meters less than the

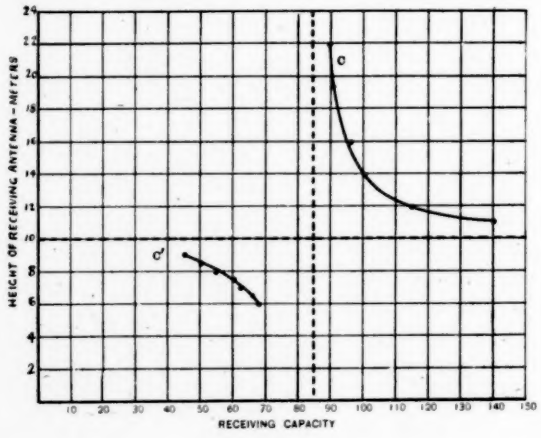

Fig. 20.

total height measured from the coil. Fourteen resonance curves similar to those of Fig. 15 were taken with various heights of the eight-wire antenna. The individual curves are not here given, but the curve obtained by plotting resonant receiving capacity against height of antenna is shown in *CC'*, Fig. 20. The equation of this curve is approximately

$$(H_e - 10)(C_4 - 85) = 60. \qquad (c)$$

Equation (*c*) is of the same form as equations (*a*) and (*b*). The three equations have approximately the same constant in the second factor. The agreement of the values calculated from equation (*c*) and those observed with the eight-wire antenna is shown in Table XIII.

TABLE XIII.

Relation Between Height of Receiving Antenna and Resonant Capacity. Eight Wires Receiving.

Meters Antenna Above Coil H_e	Resonant Capacity Observed, C_4	Resonant Capacity Calculated.	Meters Antenna Above Coil H_e	Resonant Capacity Observed, C_4	Resonant Capacity Calculated.
22	89	90	9	45	25
19	91	91.6	8.5	50	45
16	96	95	8	55	55
14	102	100	7.5	60	61
12	115	115	7	62	65
11	140	145	6.5	66	68
9.5	no resonance		6	68	70

The values of the deflections are omitted from Table XIII., because, on account of the time required for changing the height of the eight-wire antenna, the observations were extended over three or four days with various conditions of weather. While the maximum of any curve could be accurately located, a comparison of the deflections at the maxima throughout the whole series does not give reliable results. In the lower branch of the hyperbola C', Fig 20, the best height was 7 meters, with C_4 at 62, which falls about with the maxima A' and B' of Fig. 19. The corresponding maximum in the upper branch, C, of the hyperbola was not determined.

The results obtained in this experiment, IX., are:

1. By having a greater capacity in the antenna, the top branch of the hyperbola is made to run up somewhat nearer to the vertical asymptote than in the previous experiments.

2. The hyperbola CC' is found to have the same vertical asymptote as the hyperbolas AA' and B', of Fig. 16 and Fig. 17.

3. The best height of the antenna in the upper branch was not determined.

4. In the lower branch, the best height was that which had for side capacity $C_4 = 62$. This is about the value of C_4 (60 and 62)

that was resonant with the best height of antenna of four wires and of one wire.

Experiment X. One Wire Sending and Two Wires Receiving.— In the preceding experiments, Experiments VI., VII., VIII., and IX., on the relation between height of antenna and resonant capacity in the condenser circuit at the receiving station, the sending circuit has been kept constant. The period of the incident waves was thus the same in all the experiments.

In the present experiment a different period of waves is employed. The new period of waves was obtained by putting up a different antenna at the sending station and readjusting the glass condenser at the sending station. The new antenna was a single wire .208 cm. in diameter and 16 meters long, measured from the coil L_2. Instead of making the adjustment of the sending condenser circuit

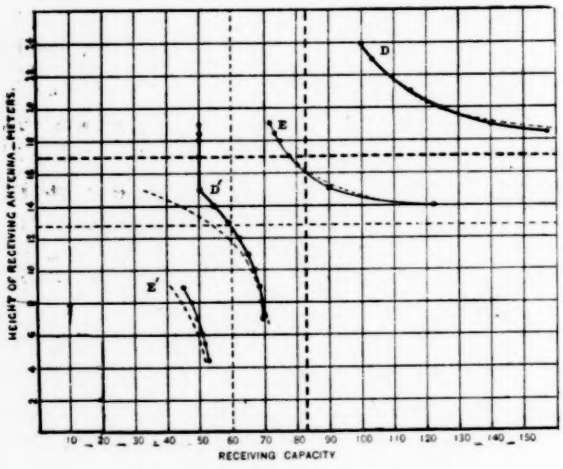

Fig. 21.

(circuit I.) to resonance with the sending mast circuit (circuit II.) by a series of curves at the receiving station as was done in the previous experiments, the adjustment was made by means of the hot-wire ammeter, A, Fig. 13, between L_2 and the ground. The capacity in circuit I. was chosen so as to give a maximum current in circuit II. as indicated by the ammeter.

With this sending station, resonance curves were taken at the receiving station with various heights of a two-wire antenna. The

length of the parallel portions of the two wires was 2.5 meters less than the whole length above the coil.

With this arrangement, the relation between height of antenna and resonant receiving capacity is shown in Fig. 21. Two hyperbola-like curves DD' and EE' were obtained. The two curves, especially the upper branches, D and E have approximately the equations

$$(H_d - 17)(C_4 - 83) = 120, \qquad (d)$$

and

$$(H_e - 12.8)(C_4 - 60) = 75, \qquad (e)$$

respectively.

The calculated values for the two curves and the position of their asymptotes are plotted as the broken lines in Fig. 21. The observed values are represented along the unbroken lines. The departure of the observed values from the calculated values in the lower

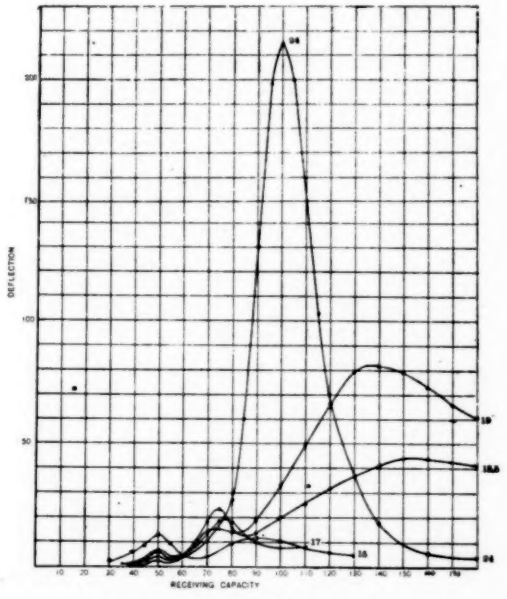

Fig. 22.

branches of the hyperbolas is considerable. Apart from the fact of the existence of two hyperbola-like curves, the results are also interesting as showing how the lower branch of the curves (*e. g.*, D') arises. In this connection see also Fig. 22. In Fig. 22 five curves

are given, in which deflections are plotted against receiving capacity, for various lengths of the two-wire antenna between 24 meters and 15 meters, as indicated by the numbers at the right-hand termination of the curves. All of these curves show a small maximum at 50, a maximum in the region between 70 and 80, and a third maximum between 100 and 160. As wire is cut off from the receiving antenna, the maximum at 50 remains fixed in position but grows in amplitude; the second maximum shifts from 73 to 75, increasing in amplitude, then shifts to 77 and 90 decreasing in amplitude. The third maximum, as wire is cut off, progresses from 100 to 135 and then to 155, always decreasing in amplitude. As wire is further cut off below the lengths for which the curves in Fig. 22 are plotted, the second and third maxima disappear, and the maximum at 50 moves by steps over to 70. When the length of antenna is about 9 meters still another maximum begins to be evident in the neighborhood of 45. There is no harmonic relation among these maxima. By plotting a large number of curves similar to those of Fig. 22, and locating all the maxima, data were obtained for the hyperbolas *DD'* and *EE'* of Fig. 21.

We shall see in what follows that the presence of two hyperbolas, instead of one as in the previous experiments, is probably due to the fact that the sending station sends out two sets of waves with different periods. The presence of two distinct sets of oscillations in an electromagnetically connected oscillator has been repeatedly predicted from the theory. The interference system produced by these two sets of oscillations may be seen in Fig. 23, which is a rotating mirror photograph of a spark gap in the mast circuit between the coil L_2 and the ground at the sending station. After a few oscillations the discharge of one period is damped out, and the oscillation of the other period alone continues for twenty or thirty vibrations further.

These two periods at the sending station give rise to the two hyperbolas at the receiving station. The reason that only one hyperbola was observed in the former experiments is found perhaps in the fact that the other hyperbola would have been discovered only with resonant capacities greater than those at my disposal. This explanation presupposes that the hyperbolas obtained in Experiments VI.

to IX. were in every case due to the oscillation of higher frequency, and that this is the weaker of the two oscillations at the sending station. Consistent with this, in Experiment X. the hyper-

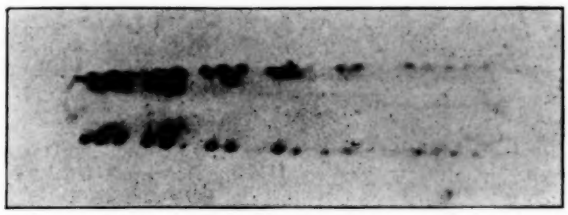

Fig. 23.

bola DD' must be supposed due to the more persistent oscillation, and it must be supposed that this is of lower frequency.

The revolving mirror picture does not show whether it is the oscillation of higher frequency or that of lower frequency that is more persistent. The only experimental evidence of the consistency of the above explanation is found in the fact that the best deflections in the former experiments, VI. to IX., were only about one fifth as large as the best deflections in experiment X., although the readings of the ammeter at the sending station were about the same in the two cases.

VIII. Theoretical Discussion of Results.

Theory. — The results of the above experiments agree approximately with deductions from the elementary theory of alternating currents.

If we have have two circuits, III. and IV., with resistances, self-inductances and localized capacities, and we suppose a sinusoidal electromotive force, $E_0 \sin \omega t$, impressed on Circuit III., the currents i_3 and i_4 in the two circuits obey the differential equations

$$L_3 \frac{di_3}{dt} + M \frac{di_4}{dt} + R_3 i_3 + \frac{\int i_3 dt}{C_3} = E_0 \sin \omega t, \qquad (1)$$

and

$$L_4 \frac{di_4}{dt} + M \frac{di_3}{dt} + R_4 i_4 + \frac{\int i_4 dt}{C_4} = 0, \qquad (2)$$

in which M is the mutual induction of the two circuits.

The solution[1] of these two equations as simultaneous gives for the amplitude of current in Circuit IV. the value

$$I_4 = \frac{E_0 M\omega^3}{\left[\left\{(L_3L_4 - M^2)\omega^4 - \left(\frac{L_3}{C_4} + \frac{L_4}{C_3} + R_3R_4\right)\omega^2 + \frac{1}{C_3C_4}\right\}^2 + \left\{-(L_3R_4 + L_4R_3)\omega^3 + \left(\frac{R_3}{C_4} + \frac{R_4}{C_3}\right)\omega\right\}^2\right]^{\frac{1}{2}}}. \quad (3)$$

For a given value of ω, I_4 is a maximum when

$$(L_3L_4 - M^2)\omega^4 - \left(\frac{L_3}{C_4} + \frac{L_4}{C_3}\right)\omega^2 = \frac{1}{C_3C_4} = 0. \quad (4)$$

By transposing and factoring, this equation may be written

$$\left(\frac{1}{C_3} - L_3\omega^2\right)\left(\frac{1}{C_4} - L_4\omega^2\right) = M^2\omega^4.$$

If we let $a = L_3L_4 - M^2$, we obtain

$$\left(C_3 - \frac{L_4}{a\omega^2}\right)\left(C_4 - \frac{L_3}{a\omega^2}\right) = \frac{M^2}{a^2\omega^4}. \quad (5)$$

Equation (5) is of the form of the empirical equations (*a*), (*b*), (*c*), (*d*) and (*e*).

Tests of the Agreement of the Empirical Equations with the Theoretical Equation. — In experiments VI., VII., VIII., IX. and X., the following equations were found to give approximately the relation between height of antenna and resonant condenser capacity at the receiving station:

Experiments.	No. Wires Sending.	No. Wires Receiving.	Equation.	
VI., VIII.	Four	Four	$(H_a - 11.8)\ (C_4 - 84.6) = 88$	(*a*)
VII.	"	One	$(H_b - 30)\ (C_4 - 84.6) = 230$	(*b*)
IX.	"	Eight	$(H_c - 10)\ (C_4 - 85) = 60$	(*c*)
X.	One	Two	$(H_d - 17)\ (C_4 - 83) = 120$	(*d*)
X.	"	"	$(H_e - 12.8)\ (C_4 - 60) = 75$	(*e*)

First Test. On τ. — If in equation (5) we divide the right-hand member by the product of the constant terms of the left-hand member, we have

[1] Compare Webster's Electricity and Magnetism, p. 502.

$$\frac{M^2}{a^2\omega^4} \div \frac{L^4}{a\omega^2} \times \frac{L_3}{a\omega^2} = \frac{M^2}{L_3 L_4} = \tau^2. \qquad (6)$$

The quantity τ defined by equation (6) is called the "coefficient of coupling." A value of τ may be obtained directly from each of the empirical equations. For example, $88 \div 11.8 \times 84.6 = .296 = \tau$. The agreement of the values of τ so obtained with each other (Table XIV.) and with its value obtained by direct measurement of the inductances is the first test to which the experiments are subjected.

TABLE XIV

Equation.	τ
(a)	.296
(b)	.300
(c)	.266
(d)	.291
(e)	.312
Average	$.291 \pm 4\%$

The data for calculating τ by direct measurement of L_3, L_4 and M are as follows:

$M_{34} = 8.90 \times 10^{-5}$ Henries, by comparison with earth inductor.
$L_3 = 125 \times 10^{-5}$ " by Maxwell's Bridge.
$L_4 = 7.04 \times 10^{-5}$ " by Resonance Method.[1]

The resulting value of τ is .290 which is in good agreement with the several values of τ determined from the position of the asymptotes.

Second Test. On the Period of the Incident Waves. — From equation (5) the constant subtraction from C_4 in the empirical equations (when reduced to farads) should equal $L_3/a\omega^2$. Let us calculate the period of incident waves from each of the equations (a), (b), (c), (d), and (e). The calculation is made as follows:

From equations (5) and (a),

$$\frac{L_3}{a\omega^2} = 84.6 \text{ cm. of the air condenser}$$

[1] This method is described in the former paper: PHYSICAL REVIEW, Vol. XIX., p. 209, 1904.

$$= 84.6 \times 2.77 \times 10^{-11} \text{ farads},$$

$$a = L_3L_4 - M^2 = L_3L_4(1 - \tau^2) = .915L_3L_4,$$

$$\therefore \frac{1}{\omega^2} = .915 \times 84.6 \times 2.77 \times 7.04 \times 10^{-16},$$

$$\text{and } T = \frac{2\pi}{\omega} = 2.44 \times 10^{-6} \text{ seconds.}$$

Table XV. contains the results for the periods (T) obtained from the five empirical equations, together with the values obtained by photographing the oscillation in the mast circuit at the sending station with the aid of the revolving mirror.

TABLE XV.

Period of the Waves.

Sending Antenna.	Calculated from the Theory.		Measured by Revolving Mirror.
	Equation.	T Millionth Second.	T Millionth Second.
Four wires.	(*a*)	2.44	3.45
	(*b*)	2.44	
	(*c*)	2.45	
One wire.	(*d*)	2.52	2.47
	(*e*)	2.12	

In the examination of Table XV. it should be remembered that the T here calculated is the period of the electromotive force impressed on the mast circuit at the receiving station, and, hence, it is the period at the sending station. The curves that gave equations (*a*), (*b*) and (*c*), though taken at the receiving station with divers arrangements of circuits at the receiving station, were with the *same sending circuit*, and should give the same value of T. The agreement with each other of these three values of T, calculated from the theory, is satisfactory.

The values of T calculated by the aid of the theory from (*d*) and (*e*) are for a sending circuit different from that of (*a*), (*b*) and (*c*). *Two* values of T were obtained, corresponding to the *two* hyperbolas of Fig. 21, because an electromagnetically connected sending station, in which the coefficient of coupling is not small, has *two characteristic periods*, as has been repeatedly emphasized by various

theoretical writers, and as may be seen by reference to the spark photograph of Fig. 23.

The duplicity of the spark perhaps accounts for the inconsistency between the first three calculated values of T (2.44), and the first value obtained from the spark photographs (3.45). I could measure in the spark picture only the period that was least damped, which is the longer period. The value calculated from (*d*) is the longer period, and the value there agrees with the spark picture. The values calculated theoretically from (*a*), (*b*) and (*c*) are perhaps the shorter period, and hence do not agree with the value obtained from the spark photograph. Strong evidence that the period determining the equations (*a*), (*b*) and (*c*) was the strongly damped oscillation with the four-wire sending station, while the period entering in (*d*) was the less damped oscillation of the one-wire sending circuit, is had from a comparison of the amplitude of the curves of Fig. 15 with those of Fig. 22. The best deflection in Fig. 22 is about five times the best deflection in Fig. 15.

Third Test. Determination of the Period by Another Method. — As a further attempt to show the correctness of the period calculated by the aid of the theory from the equations (*a*), (*b*) and (*c*), another test is applied.

This test makes use of the positions of the maxima of the maxima of the curves of Fig. 15. These maxima of maxima are shown in Fig. 19.

If we substitute the condition for resonance, equation (4), in the equation of current, equation (3), we have, after squaring and dividing by ω^2,

$$I_4^2 = \frac{E_0^2 M^2 \omega^4}{R_3^2 R_4^2 \omega^2 + \left\{ -(L_3 R_4 + L_4 R_3)\omega^2 + \left(\frac{R_3}{C_4} + \frac{R_4}{C_3} \right) \right\}^2} \qquad (7)$$

This is the square of the current in Circuit IV. when IV. is in resonance with III., and is hence proportional to the maximum deflection of the several resonance curves of Fig. 15. In equation (7), C_4 and C_3 are related by the resonance condition, equation (5), from which,

$$\frac{1}{C_3} = \frac{M^2 \omega^4}{\frac{1}{C_4} - L_4 \omega^2} + L_3 \omega^2. \qquad (8)$$

Equation (7), when C_3 is replaced by its value from equation (8) should be the equation of the curves of Fig. 19.

Let us compare the maxima of Fig. 19 with the maxima obtained from equation (7).

Eliminating C_3 from (7), and putting

$$\frac{dL_4}{dC_4} = 0,$$

we have, as the condition for maximum deflection,

$$\left\{ -L_4 R_3 \omega^2 + \frac{R_3}{C_4} + \frac{R_4 M^2 \omega^4}{\frac{1}{C_4} - L_4 \omega^2} \right\} \left\{ -\frac{R_3}{C_4^2} + \frac{\frac{R_4 M^2 \omega^4}{C_4^2}}{\left(\frac{1}{C_4} - L_4 \omega^2\right)} \right\} = 0.$$

The first factor set equal zero leads to imaginary values of C_4. The second factor gives

$$\left(\frac{1}{C_4} - L_4 \omega^2\right)^2 = \frac{R_4 M^2 \omega^4}{R_3},$$

whence

$$\frac{1}{C_4'} = L_4 \omega^2 + M \omega^2 \sqrt{\frac{R_4}{R_3}}, \tag{9}$$

and

$$\frac{1}{C_4''} = L_4 \omega^2 - M \omega^2 \sqrt{\frac{R_4}{R_3}}, \tag{10}$$

C_4' and C_4'' are the two values of the capacity in circuit IV. which give the greatest deflections in the two branches of the hyperbola. These values in experiments VI., VII., and VIII., as seen by reference to Fig. 19, are

C_4'	C_4''	
95	61	Curve AA'
	62	Curve B'
	62	Note on p. 32.
Average, 95	61.6	

Adding equations (9) and (10), and recalling that 1 cm. of the air condenser equals 2.77×10^{-11} farads, we have

$$\omega^2 = \frac{1}{2L_4}\left(\frac{1}{C_4'} + \frac{1}{C_4''}\right)$$

$$= \frac{1}{2 \times 7.04 \times 2.77 \times 10^{-16}}\left(\frac{1}{95} + \frac{1}{61.6}\right).$$

Whence

$$T = \frac{2\pi}{\omega} = 2.41 \times 10^{-6} \text{ seconds.}$$

This value of T is within about one per cent. of the three values previously obtained (Table XV.) by computations from equations (a), (b) and (c), with which it should agree.

Further experiments are in progress in which it is proposed: (1) To change the coils in the mast circuits, (2) to change the mutual inductance between the mast circuits and the side circuits, (3) to investigate the conditions for resonance in the directly connected system of circuits.

JEFFERSON PHYSICAL LABORATORY,
HARVARD UNIVERSITY, CAMBRIDGE, MASS.,
December 26, 1904.

WATER OF CONSTITUTION AND OF CRYSTALLIZATION.

BY W. W. COBLENTZ.

IT is well known that many substances belonging to all classes of chemical compounds unite with water, forming combinations from which it may be expelled by the application of a greater or less degree of heat; and from the circumstance that many of these combinations are crystalline, the water is said to be present in the form of *water of crystallization*. It is generally supposed that the molecules of water exist in their entirety situated among the other molecules of the substance. "The compounds containing water of crystallization are usually denied the title of atomic compounds ordinarily applied to combinations of two or more elements in which the constituent atoms are associated in a single molecule, under the influence of the force which has received the name of chemical affinity, and in contradistinction are termed *molecular* compounds, being regarded as combinations of two or more separate molecules, *e. g.*, $CaSO_4 + 2H_2O$. This, although perhaps true of many compounds containing water of crystallization, is certainly not true of all, and notably the sulphates, which are only deprived of their water of crystallization by heating to a high temperature. The last molecule retained with such persistency was termed, by Graham, *water of constitution*, to distinguish it from water of crystallization.

"In the present state of our knowledge it is impossible absolutely to define the meaning of these terms, or even to say that there is an absolute difference between the so-called water of crystallization and water of constitution, and not merely one of degree."[1] To sum up, water of crystallization differs from combined water in that it does not belong to the molecular structure but only to the crystalline structure of the substance. Examples of substances contain-

[1] Encyclopædia Brittanica, 9th Ed., V., pp. 489 and 505.

ing water of constitution are sugar, and brucite, $Mg(OH)_2$, in which the O and the H atoms are supposed to be scattered about in the molecule, and do not unite to form H_2O until sufficient heat is applied to cause "combustion." There is considerable evidence that in many compounds the affinity is different for the different parts of the crystal water.[1] For example, ordinary alum, $AlK(SO_4)_2 + 12H_2O$, gives off five molecules of H_2O at 100°, and five molecules at 120°, while the last two molecules remain until heated to 200°. In the same manner in the sulphates of Mg, ($MgSO_4$, 7 H_2O), Zu, Fe, Ni and Co the last molecule of H_2O passes off at 200° to 300° C.

Another distinction between water of constitution and water of crystallization is that in compounds containing the former, after the water is expelled by application of heat, *e. g.*, mica, the residue *will not* take up water again, while in the latter water is taken up again. An example of the latter is copper sulphate, $CuSO_4 + 5H_2O$. On applying heat, the blue crystal becomes a crumbling white mass, which, if put in water or even permitted to stand in the air, absorbs water and resumes its blue color and crystalline structure.

We thus see that the question of the association of atoms of oxygen and of hydrogen in certain compounds is by no means a settled one. The purpose of this note is to call attention to data bearing on this subject.

The absorption spectrum of water has been found by Julius,[2] Paschen,[3] Aschkinass[4] and others. All observers agree in their location of large absorption bands at the wave-lengths 1.5 μ, 3μ, 4.7 μ and 6 μ. No one questions their being due to water. The only investigation bearing on this subject, from the standpoint of infra-red absorption spectra, is that of Königsberger,[5] who studied the pleochroism of gypsum (selenite), muscovite and biotite mica, etc. The thickness of the plate of selenite was 2.57 mm., which was sufficient to cause complete opacity at 3 μ. As a consequence he could not emphasize the importance of his observations

[1] Graham-Otto, Lehrbuch de Chemie, II., p. 173.
[2] Julius, Verhandl. Konikl. Akad. Amsterdam, Deel I., No. 1, 1892.
[3] Paschen, Ann. der Phys., 53, p. 334, 1894.
[4] Aschkinass, Ann. der Phys., 55, p. 406, 1895.
[5] Königsberger, Ann. der Physik., 61, p. 687, 1897.

as much as they deserve. He does mention, however, that from the absorption band at 1.5 μ, which coincides with that of water, it would appear that in selenite $CaSO_4 + 2H_2O$, the water of crystallization does not seem different from ordinary water.

The present investigation deals with the infra-red absorption spectra of the minerals brucite, $Mg(OH)_2$ and selenite, $CaSO_4 + 2HO$, using a mirror spectrometer, a rock salt prism, and a Nichols radiometer.[1] The former contains the O and H in the form of water of constitution, while the latter has the H_2O present as water of crystallization. The reader may ask what right we have to discuss this question from the mere study of the absorption spectra of water and a few minerals like the aforesaid. My reply is that from an extensive study of organic compounds,[2] in which it was found that certain groups of chemical compounds have similar absorption spectra, this seems permissible. Then, too, these curves, considered in connection with those of Königsberger, seem conclusive evidence, in themselves, that water of crystallization and water of constitution are distinct.

WATER, H_2O.

The transmission curves of water, found by Aschkinass (*loc. cit.*), are given in Fig. 1. They show that in order to be able to penetrate the infra-red beyond 3 μ the film must be about. 01 mm. in thickness.

The transmission curve for water, used as a comparison spectrum in the present work, is given in Fig. 3. The film was much thinner than these, and the 4.65 μ band has almost disappeared.

MICA.

The curves for the micas — muscovite, $H_4K_2Al_6Si_6O_{24}$, and biotite (contains iron) — given in Fig. 2, *b* and *c*, are due to Königsberger (*loc. cit.*).

The curves are so entirely different from those of water that one does not hesitate to say that the condition of the oxygen and the hydrogen atoms, which are scattered about in the molecule, combining to form a molecule of water on applying heat, is entirely different from the molecule of the water of crystallization in selenite.

The 2.9 μ band of muscovite mica, which lies close to the water band at 3 μ, can hardly be attributed to water for the simple reason

[1] Described in the PHYSICAL REVIEW, Vols. XVI. and XVII., 1903.

[2] Prelim. Report in Astrophys. Jour., 20, p. 207, 1904.

that there is no band at 1.5 μ. Then, too, the general transparency is entirely different.

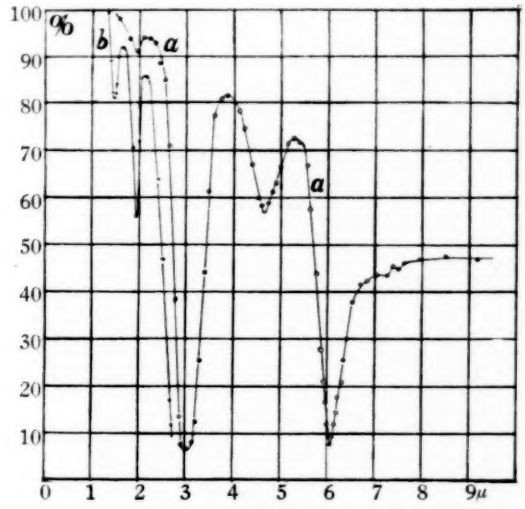

Fig. 1.

Transmission Curves for Water. Thickness of film: Curve $a = .01$ mm.; curve $b = .05$ mm.

The absorption of the biotite, in the visible spectrum, is due to the dark coloring matter, iron oxide, which it contains.

Brucite, $Mg(OH)_2$.

For this mineral as well as the selenite I am indebted to Professor Gill, of the Department of Mineralogy. Brucite is of especial interest on account of its OH-groups. It is also an illustration of water of constitution. On application of considerable heat the O and H atoms combine into, and pass off as a molecule of water, leaving as a residue the white MgO.

The brucite curve, *a*, Fig. 2, has but one large absorption band, at 2.5 μ, beyond which there are no bands until we arrive at 9 μ, beyond which point there is complete opacity.

This is but the second substance discovered which has an absorption band near the visible spectrum. The first one, beryl, $H_2Be_6Al_4Si_{12}O_{37}$, which has an enormous absorption band at .86 μ, was found by Königsberger (*loc. cit.*).

As a whole the curve of brucite shows that the OH-group, to the resonance of which numerous phenomena have been ascribed,

by different investigators, cannot be very instrumental in causing the absorption bands of water. Also, from its dissimilarity with the water curve, one would infer that water of constitution is a different condition from water of crystallization.

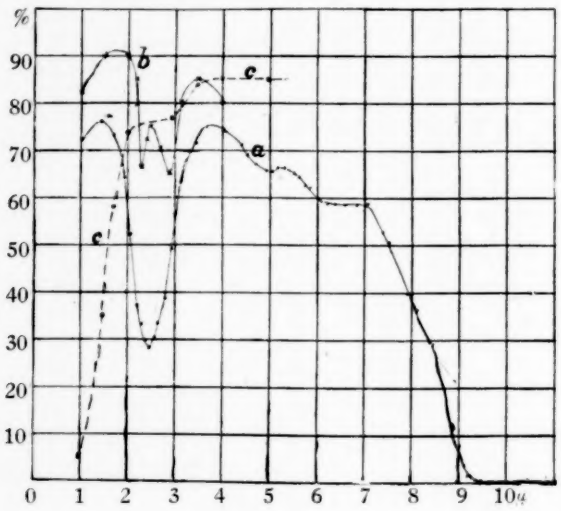

Fig. 2.

Curves a = brucite; b and c = mica.

SELENITE, $CaSO_4 + 2H_2O$.

Because of the great opacity of water, it is difficult to explore the absorption spectra of substances containing water of crystallization, unless the plates are very thin. Fortunately selenite can be split into folia of almost any desired thickness.

In the present work the selenite was of the massive variety which yielded thin, highly polished folia. In Fig. 3, curve a is due to Königsberger who used a plate 2.57 mm. in thickness, which prevented him from exploring the spectrum beyond 3 μ.

In the present work the plate was only .126 mm. in thickness and gave a very satisfactory curve, b. The absorption bands, at 1.5 μ, 2.95 μ, 4.6 μ and 6 μ, as well as the general absorption show a close resemblance to that of water. The band at 6 μ is somewhat obliterated by the increasing opacity of the plate which becomes opaque at 8 μ. Here the band of metallic absorption (reflection) begins, the maximum of which as determined by Aschkinass [1]

[1] Aschkinass, Ann. der Phys., 1, p. 42, 1900.

occurs at 8.69 μ. The comparison spectrum of water is given in curve *c*, which is for a film of water between fluorite plates. The plates were pressed together as close as possible in order to produce an unusually thin film. From the interference fringes, after the

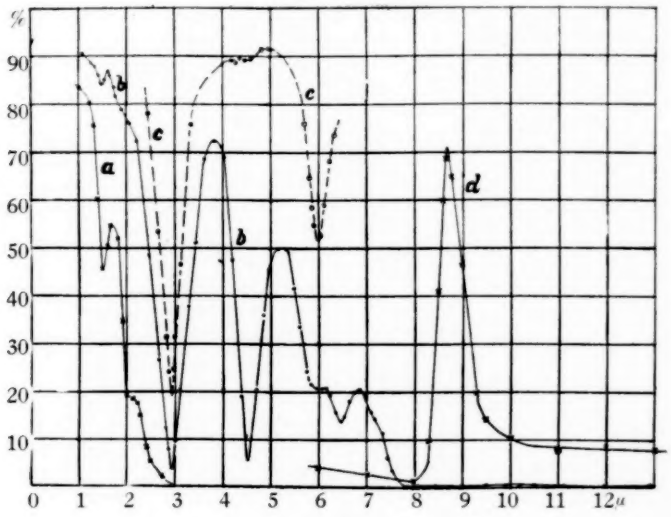

Fig. 3.

Curves *a*, *b* and *d* = selenite; *c* = water.

water had evaporated, the thickness of the film was estimated to be about .001 mm. This accounts for the absence of the small band at 4.65 μ, see Fig. 3.

The band at 6.5 μ coincides with the one for water vapor, found by Paschen at 6.5 μ.

The computed thickness of the layer of water contained in the .126 mm. plate of selenite is .059 mm., which explains the greater depth of the 4.65 μ band, and the greater opacity as compared with the water curve. The plate 2.57 mm. in thickness, curve *a*, represents a layer of water .12 mm. in thickness. We thus see a striking similarity between the curves for water and selenite.

The method of selective reflection seems better adapted for studying minerals far in the infra-red, especially compounds like alum, which has 12 molecules of water. In fact it seems the proper method for making an extensive study of minerals containing water

of crystallization, since, with but few exceptions, they cannot easily be obtained in thin films.

As already indicated, the unusual similarity of the curve of selenite, and the great dissimilarity of the brucite and mica curves, to that of water, indicates that water of crystallization is not different from ordinary water. The whole shows that the difference between the so-called water of constitution and water of crystallization is more than "merely one of degree."

PHYSICAL LABORATORY, CORNELL UNIVERSITY,
December 5, 1904.

THERMODYNAMICAL POTENTIALS.

By W. P. Boynton.

THE following paper is an attempt to give definiteness to our notions of the thermodynamical potentials of Professor Gibbs by deducing them for a substance following van der Waals' equation. In a former paper[1] this has been done for the Intrinsic Energy of the substance.

Using the notation and, so far as they are applicable, the results of that paper, the four potentials are defined by the four differential equations

$$dU = TdS - pdv, \tag{1}$$

$$d\psi = -(pdv + SdT), \tag{2}$$

$$d\zeta = vdp - SdT, \tag{3}$$

$$d\chi = Td - Svdp, \tag{4}$$

which lead to the three symbolic integral equations

$$\psi = U - TS, \tag{5}$$

$$\zeta = U - TS + pv, \tag{6}$$

$$\chi = U + pv. \tag{7}$$

To apply these to van der Waals' equation

$$(p + a/v^2)(v - b) = RT, \tag{8}$$

we need the expressions for the total energy and entropy of such a substance. On the assumption that the specific heat at constant volume is a constant, these are[2]

$$U = \int_0^T C_v dT + \int_\infty^v \frac{a}{v^2} dv = C_v T = \frac{a}{v}, \tag{9}$$

$$S = C_v \log T + R \log (v - b) + \text{Const.} \tag{10}$$

[1] Phys. Rev., XI., pp. 291–303, 1900.

[2] In the former paper, p. 292, the second integral was wrongly given the lower limit b. The correct result is also given by taking the integral $U = \int_v^\infty p dv$ underneath an isentropic.

The remainder of the work consists of the elimination between equations (5) to (10) inclusive of the quantities not found in equations (1) to (4) respectively, and the simplification and discussion of the four equations thus derived.

The simplification is attained by the use of the so-called reduced coördinates, and the assumption that the critical state is the standard condition, so that its data provide the constants of integration. Introducing these coördinates and indicating values corresponding to the critical state by the subscript c, the expression for the entropy becomes

$$S - S_c = C_v \log \vartheta + R \log \frac{3\varphi - 1}{2}, \tag{11}$$

$$\frac{S - S_c}{C_v} = \log \vartheta + \frac{R}{C_v} \log \frac{3\varphi - 1}{2}, \tag{12}$$

$$\vartheta = e^{\frac{S - S_c}{C_v}} \left(\frac{3\varphi - 1}{2}\right)^{-R/C_v}. \tag{13}$$

Van der Waals' equation itself, in "reduced" form is also given for convenience

$$\left(\pi + \frac{3}{\varphi^2}\right)(3\varphi - 1) = 8\vartheta \tag{14}$$

and the relations

$$p = \frac{1}{27}\frac{a}{b^2}\pi, \; v = 3b\varphi, \; T = \frac{8}{27}\frac{a}{bR}\vartheta. \tag{15}$$

U.

Eliminating between equations (9), (13) and (15),

$$U = \frac{a}{b}\left[\frac{8}{27}\frac{C_v}{R} e^{(S - S_c)/C_v} \left(\frac{3\varphi - 1}{2}\right)^{-R/C_v} - \frac{1}{3\varphi}\right], \tag{16}$$

$$U_c = \frac{a}{b}\left[\frac{8}{27}\frac{C_v}{R} - \frac{1}{3}\right]. \tag{17}$$

So that for the value $R/C_v = 2/3$, which holds for monatomic gases, we have the simple numerical equation

$$U/U_c = 4e^{(S - S_c)/C_v} \left(\frac{3\varphi - 1}{2}\right)^{-2/3} - 3/\varphi. \tag{18}$$

This equation differs only in numerical constants from that given in the other paper,[1] and the surface corresponding will be of the general form there shown; reference is made to that paper for further discussion.

[1] *Loc. cit.*, p. 295.

ψ.

This function sometimes indicated also by the symbol F, has been called by Helmholtz the *free energy*, and by Duhem the *internal thermodynamic potential;* and is also sometimes known as the *thermodynamic potential at constant-volume*. Eliminating between equations (5), (9), (11), and (15)

$$(19)\quad \psi = U - T(S - S_c)$$

$$= C_v T - \frac{a}{v} - C_v T \log \vartheta - RT \log \frac{3\varphi - 1}{2}$$

$$= \frac{8}{27}\frac{a}{b}\frac{C_v}{R}\vartheta - \frac{a}{3b\varphi} - \frac{8}{27}\frac{a}{b}\frac{C_v}{R}\vartheta \log \vartheta - \frac{8}{27}\frac{a}{b}\frac{R}{R}\vartheta \log \frac{3\varphi - 1}{2}$$

$$(20)\qquad \psi_c = \frac{8}{27}\frac{a}{b}\frac{C_v}{R} - \frac{a}{3b} = \frac{a}{b}\left[\frac{8}{27}\frac{C_v}{R} - \frac{1}{3}\right]$$

and the value $C_v/R = 3/2$ gives the numerical equation

$$(21)\qquad \frac{\psi}{\psi_c} = 4\vartheta(1 - \log \vartheta) - \frac{8}{3}\vartheta \log \frac{3\varphi - 1}{2} - \frac{3}{\varphi}$$

The form of the surface can be judged from Fig. 1, which gives the values of ψ/ψ_c as a function of φ for values of $\vartheta = 0,\ 1/2,\ 1,\ 2,\ 3$ respectively. As appears immediately from equation (2), $-\partial\psi/\partial v = p$ and differentiation will show that $\partial/\partial\varphi\ (\psi/\psi_c)$ is a constant multiple of π. The curves of constant pressure are easiest plotted directly from equation (14) solved for ϑ.

$$(22)\qquad \vartheta = \frac{1}{8}\left(\pi + \frac{3}{\varphi^2}\right)(3\varphi - 1)$$

Similarly $-\partial\psi/\partial T = S$ and $\partial/\partial\vartheta\ (\psi/\psi_c)$ gives a constant multiple of the entropy. The isentropics are easiest plotted from

$$(13)\qquad \vartheta = e^{\frac{S - S_c}{C_v}}\left(\frac{3\varphi - 1}{2}\right)^{-\frac{R}{C_v}}$$

Fig. 2 gives the curves of constant pressure in solid lines, and of constant entropy, for $R/C_v = 2/3$, in dotted lines, as projected upon the plane of temperature and volume. It is noteworthy that this is identical with the projection of the corresponding lines on the same plane for the surface corresponding to the original equation of van der Waals.

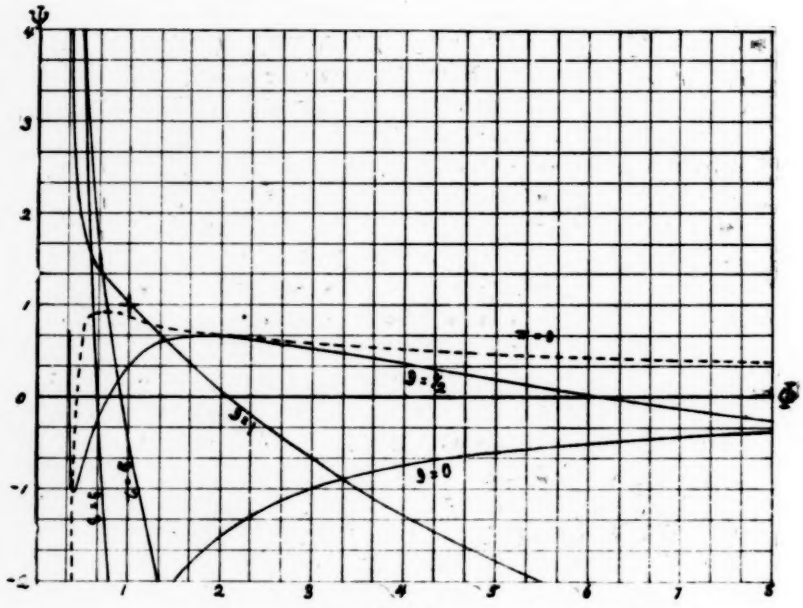

Fig. 1.

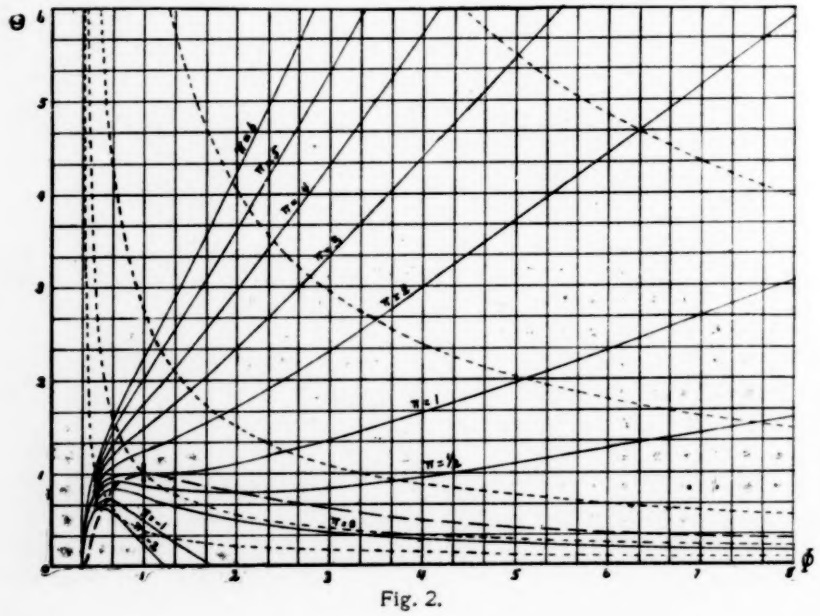

Fig. 2.

If Fig. 1 be regarded as vertical and Fig. 2 as horizontal, the model represented has evidently at the left hand a "trough," extending from the point $(\frac{1}{3}, -9)$, the intersection of the two branches of the curve $\vartheta = 0$, toward but not to, the critical point (1, 1, 1) and a "crest" running to the right, the bottom of the trough and the top of the crest being given by $\partial\psi/\partial\varphi = 0$, that is $\pi = 0$, which is given in Fig. 1 by the dotted curve.

ζ.

This function is also designated by the symbol Φ, and called *the thermodynamic potential* or, the *thermodynamic potential at constant pressure*. Its equation is obtained from (6), (9), (11) and (15).

$$(23)\quad \zeta = U - T(S - S_c) + pv$$

$$= C_v T - \frac{a}{v} - C_v T \log \vartheta - RT \log \frac{3\varphi - 1}{2} + pv$$

$$= \frac{a}{b}\left[\frac{8}{27}\frac{C_v}{R}\vartheta(1 - \log\vartheta) - \frac{1}{3\varphi} - \frac{8}{27}\vartheta \log \frac{3\varphi - 1}{2} + \frac{1}{9}\pi\varphi\right]$$

$$(24)\quad \zeta_c = \frac{a}{b}\left[\frac{8}{27}\frac{C_v}{R} - \frac{1}{3} + \frac{1}{9}\right] = \frac{a}{b}\left[\frac{8}{27}\frac{C_v}{R} - \frac{2}{9}\right]$$

It is not possible to complete the algebraic elimination of φ from this equation, since this would involve the solution of the cubic

$$(14)\qquad (\pi + 3/\varphi^2)(3\varphi - 1) = 8\vartheta.$$

Taking as usual $R/C_v = 2/3$ we obtain the numerical equation

$$(25)\quad \zeta/\zeta_c = 2\vartheta(1 - \log\vartheta) - \frac{4}{3}\vartheta \log \frac{3\varphi - 1}{2} - \frac{3}{2\varphi} + \frac{1}{2}\pi\varphi.$$

In plotting the curves of this equation values of φ corresponding to assumed values of ϑ and π were taken from curves previously plotted, except in regions of particular interest, in which values of π and φ were assumed, and ϑ computed, the results being substituted in equation (25). The analytical discussion of the curves has not been attempted, but from the figure they appear to fall into three classes: for pressure exceeding the critical they have a form not unlike that of a parabola with the vertex up; for zero and negative pressures one branch of the curve is bent sharply to the left, this point of bending corresponding to the maximum of the cor-

responding curve of Fig. 2; while for intermediate pressures of which only one example is given, namely $\pi = 1/2$, there are two sharp bends, one corresponding to the maximum, the other to the minimum of the analogous curve in Fig. 2, while the intersection of the two branches appears to determine the temperature of the saturated vapor of that pressure, the upper negative area of the loop

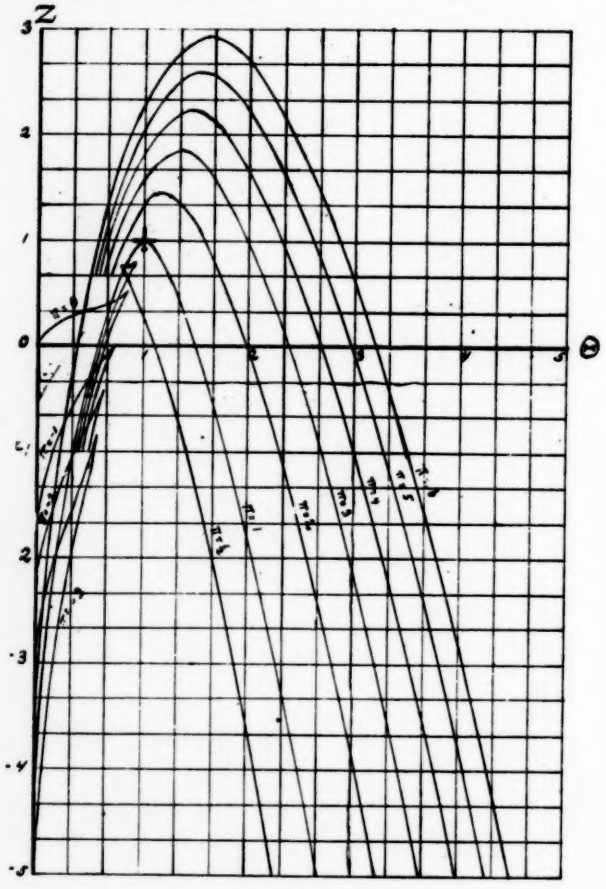

Fig. 3.

corresponding to the non-existent or imaginary transition-states of the *continuous* transformation from liquid to gas. The real or positive area below is not unlike that for pressures above the critical.

In Fig. 4 are shown the projections of the isovoluminals as solid lines and the isentropics as broken lines on the plane of pressure

and temperature. This like Fig. 2 equally represents the projection in this plane of the corresponding lines in the model of van der

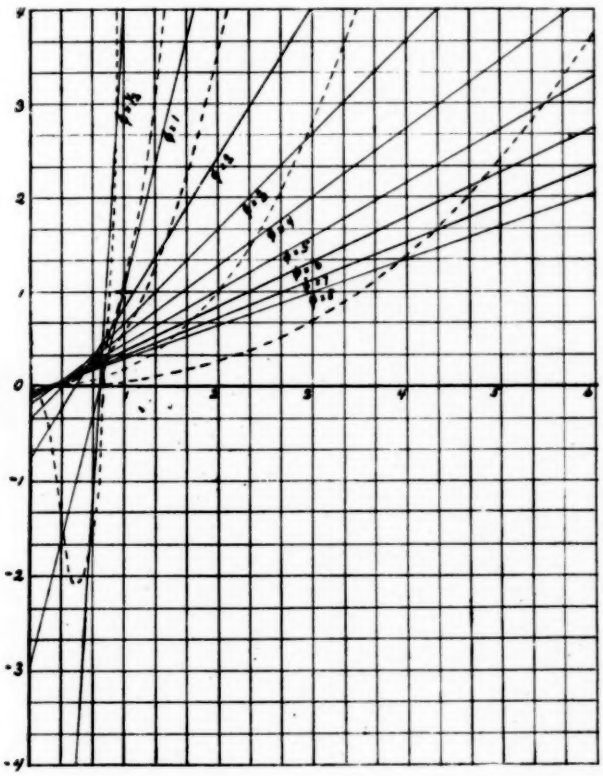

Fig. 4.

Waals' equation. It is noticeable that the isovoluminals are straight lines, and that the isentropics corresponding to an entropy equal to or greater than that of the critical stage are entirely in the region of positive pressure and exhibit no reversal of curvature, while the one isentropic drawn of less entropy has a very marked dip and reversal of curvature.

χ.

This function might be referred to as a total isentropic potential. By combining equations (7), (9), (13), and (15) we obtain

$$(26) \qquad \chi = \frac{a}{b}\left[\frac{8}{27}\frac{C_v}{R}e^{\frac{s-s_c}{C_v}}\left(\frac{3\varphi-1}{2}\right)^{-\frac{R}{C_v}} - \frac{1}{3\varphi} + \tfrac{1}{9}\pi\varphi\right]$$

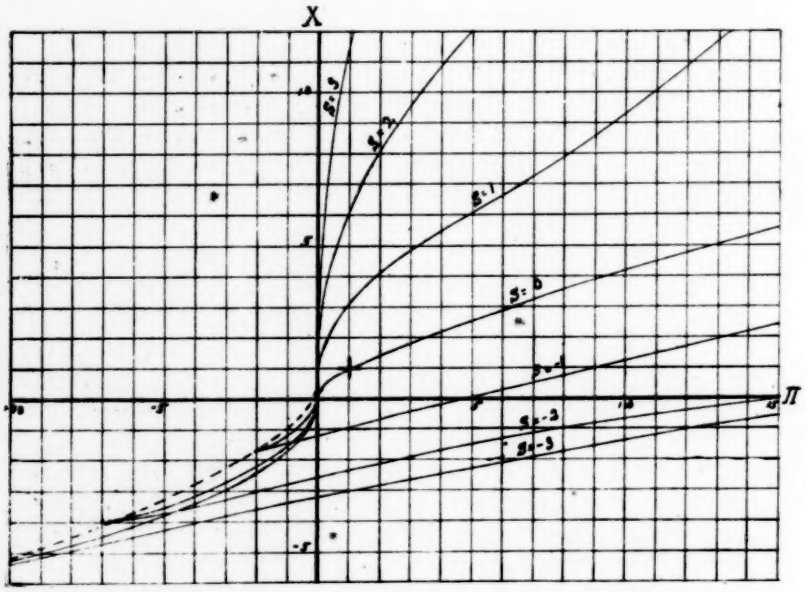

Fig. 5.

and

$$\chi_c = \frac{a}{b}\left[\frac{8}{27}\frac{C_v}{R} - \tfrac{1}{3} + \tfrac{1}{9}\right] = \frac{a}{b}\left[\frac{8}{27}\frac{C_v}{R} - \tfrac{2}{9}\right]. \tag{27}$$

It appears impossible to effect the elimination of φ, but introducing the value $R/C_v = 2/3$, we obtain the numerical equation

$$\chi/\chi_c = 2e^{\frac{S-S_0}{C_v}}\left(\frac{3\varphi - 1}{2}\right)^{-\frac{1}{3}} - \frac{3}{2\varphi} + \tfrac{1}{2}\pi\varphi. \tag{28}$$

In Fig. 5 are shown the curves of χ/χ_c for constant values of the entropy. The coördinates were computed independently from assumed values of φ and S. This figure is plotted on a scale one-third as great as that of the other figures, in order to exhibit better the portion of the curves in the third quadrant; these seem to come down to a sharp corner at the left, that is, the surface of the model would show a deep cave under a cleft, the inner edge of the cave being shown by the broken line, which corresponds to the minimum values of S in the $p - u$ diagram, the curve

$$\pi = \frac{3(\varphi - 2)}{5\varphi^3}.$$

In Fig. 6 are shown the projections on the plane of pressure and entropy of the lines of constant volume, solid, and of constant temperature, dotted. The "inner edge of the cave" is shown in the third quadrant as a heavy broken line, on which the isovoluminals and the isothermals of temperatures considerably below the critical pass from one sheet of the surface to the other. The positions of

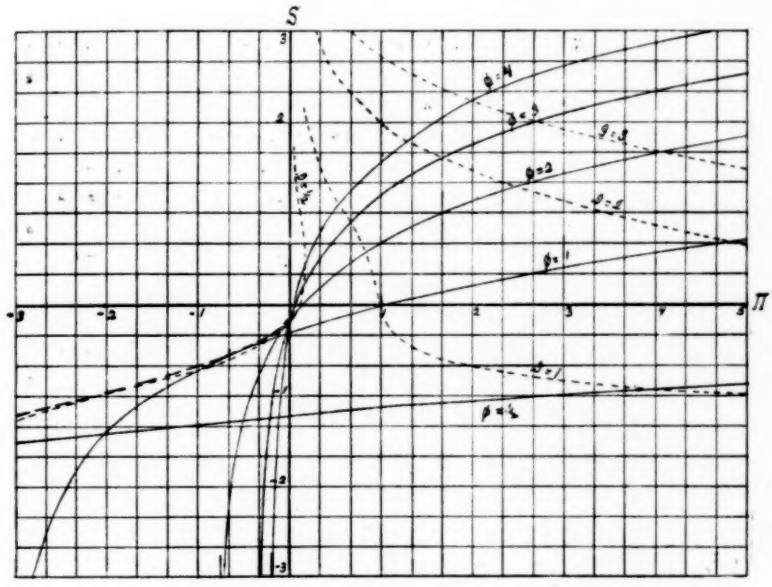

Fig. 6.

the asymptotes of the isovoluminals in the third quadrant are indicated by short vertical lines.

A complete set of models would include one for van der Waals' equation itself, and one for each of the four potentials here discussed. The curves given in this paper and its predecessor are sufficient for determining both the form of the surfaces and the location of principal lines upon them. Such a model for the function U was actually constructed for the writer by some of his pupils in the University of California.

UNIVERSITY OF OREGON.

November 11, 1904.

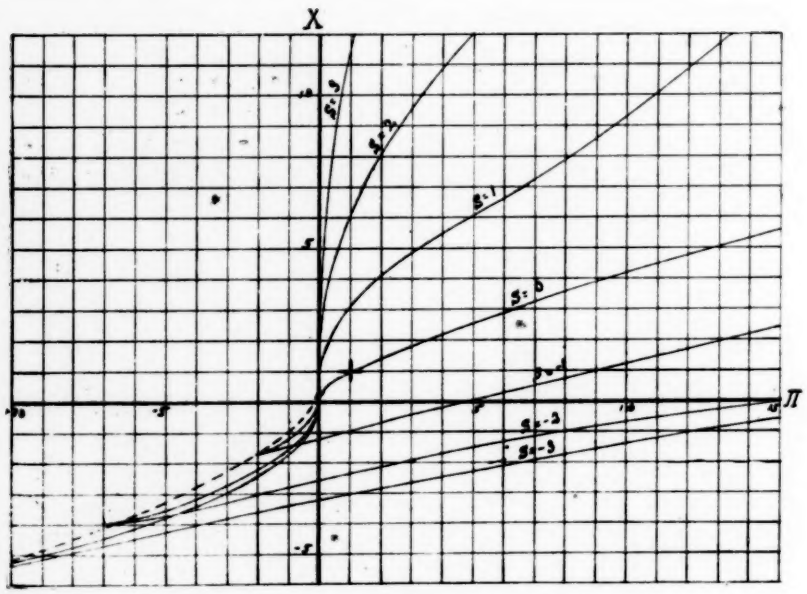

Fig. 5.

and

$$\chi_c = \frac{a}{b}\left[\frac{8}{27}\frac{C_v}{R} - \tfrac{1}{3} + \tfrac{1}{9}\right] = \frac{a}{b}\left[\frac{8}{27}\frac{C_v}{R} - \tfrac{2}{9}\right]. \tag{27}$$

It appears impossible to effect the elimination of φ, but introducing the value $R/C_v = 2/3$, we obtain the numerical equation

$$\chi/\chi_c = 2e^{\frac{S-S_c}{C_v}}\left(\frac{3\varphi - 1}{2}\right)^{-\frac{1}{3}} - \frac{3}{2\varphi} + \tfrac{1}{2}\pi\varphi. \tag{28}$$

In Fig. 5 are shown the curves of χ/χ_c for constant values of the entropy. The coördinates were computed independently from assumed values of φ and S. This figure is plotted on a scale one-third as great as that of the other figures, in order to exhibit better the portion of the curves in the third quadrant; these seem to come down to a sharp corner at the left, that is, the surface of the model would show a deep cave under a cleft, the inner edge of the cave being shown by the broken line, which corresponds to the minimum values of S in the $p - u$ diagram, the curve

$$\pi = \frac{3(\varphi - 2)}{5\varphi^3}.$$

In Fig. 6 are shown the projections on the plane of pressure and entropy of the lines of constant volume, solid, and of constant temperature, dotted. The "inner edge of the cave" is shown in the third quadrant as a heavy broken line, on which the isovoluminals and the isothermals of temperatures considerably below the critical pass from one sheet of the surface to the other. The positions of

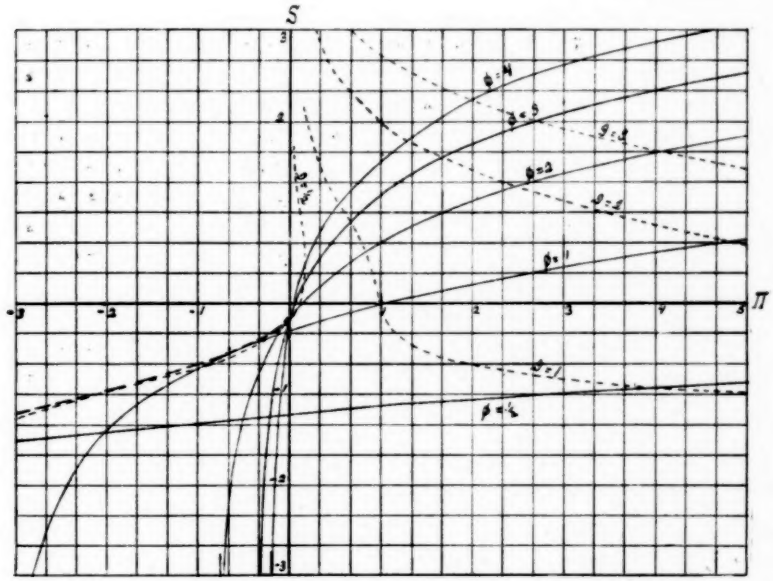

Fig. 6.

the asymptotes of the isovoluminals in the third quadrant are indicated by short vertical lines.

A complete set of models would include one for van der Waals' equation itself, and one for each of the four potentials here discussed. The curves given in this paper and its predecessor are sufficient for determining both the form of the surfaces and the location of principal lines upon them. Such a model for the function U was actually constructed for the writer by some of his pupils in the University of California.

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THE TUNING OF THERMOELECTRIC RECEIVERS FOR ELECTRIC WAVES.

BY A. D. COLE.

IN some earlier work with electric waves,[1] the writer used a modified form of Righi exciter, the two central spheres of the usual form being replaced by thin cylinders with rounded ends. In connection with this work two assumptions were made: first, that exciter and receiver were identical in wave-length, being practically identical in form and dimensions. Second, that Poincarè and Righi are right in regarding the wave-length of a linear receiver as twice the length of the receiver itself. The purpose of the present experiments was to test these two assumptions.

Two exciters were used; in one the radiating cylinders were each 37 mm. long and 3.1 mm. in diameter, in the other 25 mm. long and of the same diameter. Thus the length of the active parts of the two exciters were 74 and 50 mm. respectively, the separating oil gap being of course only a small fraction of a millimeter long. The receiver was made of the same form and diameter as the exciters, but its length was variable and a thermo-junction of fine iron and constantin wires replaced the spark gap. The details of construction were similar to those described in the earlier papers referred to. The coarser German-silver or nickelin wires of the earlier instruments were replaced by a fine wire of constantin, only .025 mm. in diameter. (The thermo-electric powers of these three metals with respect to iron are 20, 39 and 51 microvolts per C° respectively.)

The same kind of iron wire as was used before was again employed, .024 mm. in diameter.

With this receiver deflections of 300 to 400 mm. were obtained with a galvanometer of 5 ohms resistance, the figure of merit being 4.7×10^{-9} with 13 second swing.

[1] Cole, Wied. Ann., 57, 298, or PHYS. REV., 4, 54, 1896; 7, 225, 1898.

The variation in the length of the receiver was secured by sliding tubes of thin metal over its cylindrical ends. Four pairs of tubes of different lengths make possible any length of receiver between 39 and 90 mm. To facilitate these adjustments of the receiver, it was mounted behind a cylindrical lens instead of being placed at the focus of a parabolic mirror in the usual way. This lens consisted of a 2.5 liter bottle filled with benzine. Its efficiency in concentrating the radiation was such that about five times as large a deflection was obtained with it as without it.

It was expected that a maximum effect would be obtained in the receiver when its length was only slightly greater than that of the exciter, on account of their identity of form. Such however was not the case, but instead the maximum was secured when the receiver was about 20 per cent. longer than the exciter. The results of a typical case are plotted in the accompanying curve (Fig. 1) in which receiver lengths are abscissas and galvanometer deflections ordinates. An exciter was used whose total length was 0 mm., the receiver length being changed by steps of about 10 mm. from 39 to 81 mm. Four readings were taken with each length and then the whole series taken in reversed order. A distinct though not very sharp maximum occurs at 62 mm. With the receiver length equal to that of the exciter (50 mm.) the galvanometer deflections were about 10 per cent. smaller.

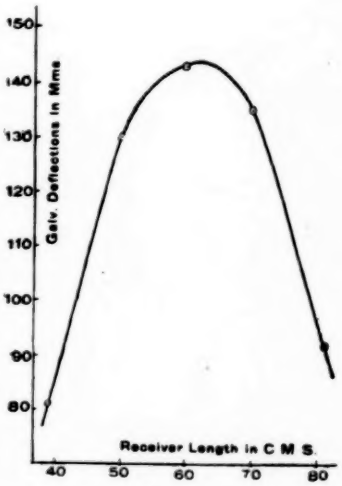

Fig. 1.

The wave-lengths were measured by three interference methods which gave results in fairly good agreement: (1) by reflection on a plane mirror behind the receiver as in Hertz's original measurements; (2) by reflection on a plane mirror behind the exciter, and (3) by Boltzmann's two mirror method.

The general character of the curves obtained is shown by Fig. 2. In this case a plane mirror behind the exciter was used. Abscissas

show the distance of the mirror behind the exciter in cm., ordinates galvanometer deflections in mm.

Maxima occur at 15.0, 23.6, 32.0.
Minima " " 9.0, 18.5, 28.0 and 37.5.
Length of intervals between max. 9.6 and 9.4.
" " " " min. 9.5, 9.5 and 9.5.
Mean interval 9.5, $\therefore \lambda = 190$ mm.

Very different interpretations of such curves have been given by different workers. Thus Hertz[1] and Zehnder[2] regard them as a measure of the wave-length of the exciter, Sarasin and de la Rive[3]

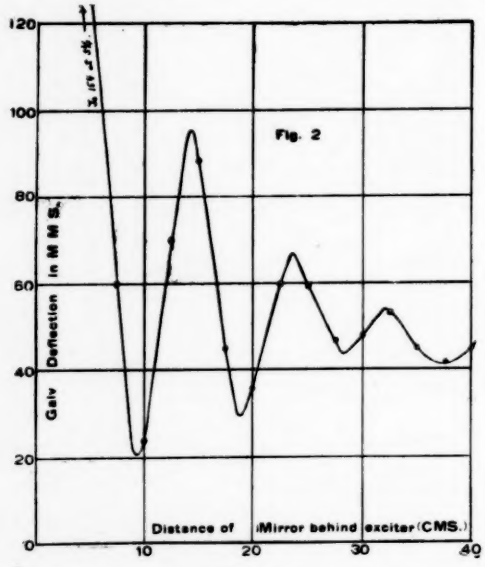

Fig. 2.

and Klemenčič and Czermak[4] have shown that the positions of the maxima and minima depend upon the dimensions of the receiver. Hull[5] concludes that they depend on both receiver and exciter and Willard and Woodman[6] reach a similar conclusion. The differ-

[1] H. Hertz, Wied. Ann., 34, 610, '88.
[2] L. Zehnder, Wied. Ann., 52, 34, '94.
[3] Sarasin and de la Rive, C. R., 110, 75, '90, and Phil. Mag., 31, 289.
[4] Klemenčič u. Czermak, Wied. Ann., 50, 174, '93.
[5] G. F. Hull, Phys. Rev., 5, 16, '97
[6] Willard and Woodman, Phys. Rev., 18, 19.

ent results obtained by these and other observers are in part due to the different kinds of exciters and receivers used and the different ways of producing interference.

With our apparatus the results clearly show that the wave-length measured is that of the receiver. For instance with exciter 74 mm. long and receiver 75, λ measured 188 mm. The experiment was repeated with the receiver length changed; λ changed correspondingly. The original receiver-length was then restored and the exciter-length changed; approximately the original wave-length was again obtained. The following are the figures:

	Exciter Length.	Receiver Length.	λ
Exp. 1	74 mm.	75 mm.	188 mm.
" 2	74	83	220
" 3	50	75	184

In these experiments the interference was produced by a plane mirror behind the receiver. That essentially the same curve is ob-

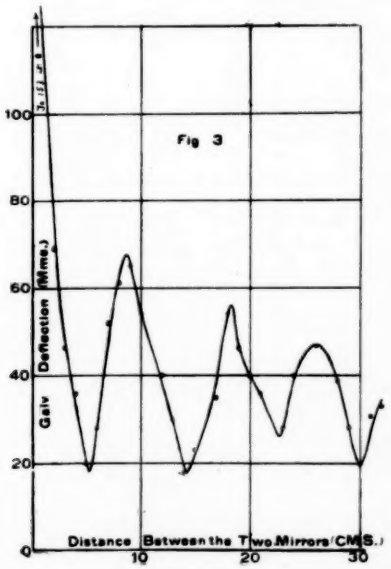

Fig. 3.

tained when the mirror is placed behind the exciter is shown by comparing the data of Exp. 1 with those given in the curve (Fig. 2).

		Mean.
Mirror behind exciter; intervals . . .	9.6–9.4– 9.5–9.5–9.5	9.5
" " receiver; " . . .	10.9–8.9–10.0–9.0–9.2	9.4

Here again the measured wave-length is evidently determined by the dimensions of the receiver. Fig. 3 shows a curve obtained by the Boltzmann two-mirror method. Two parabolic mirrors were used in this case.

Conclusions:

1. When a thermal receiver of linear form is used in connection with a Righi exciter, interference curves show the wave-length characteristic of the receiver.

2. When the receiver length is changed until a distinct maximum effect is produced from a given exciter, it is probable that the fundamental wave-length of each is the same, and an interference experiment will then (and only then) give the wave-length of the exciter.

3. The ratio of measured wave-length to receiver length as shown in five measurements (four referred to above and one other) is 2.53, 2.51, 2.65, 2.45 and 2.47 respectively. Mean 2.52. This agrees well with the value determined theoretically by MacDonald[1] (2.53) and not with the value (2.0) given by Poincarè.[2]

The above experiments were performed during the past summer at Ryerson Laboratory, University of Chicago, and the author is much indebted to Professor R. A. Millikan for placing the resources of the laboratory at his disposal and for his personal interest in the work.

OHIO STATE UNIVERSITY,
COLUMBUS, O., December, 1904.

[1] MacDonald, "Electric Waves" (Adams Prize Essay).
[2] Poincarè, "Les Oscillations Electriques."